【Article】

Non-Eulerian Inviscid Vortices

TAKAHASHI Koichi

Abstract : It is shown that taking the limit of vanishing viscosity in the Navier-Stokes equations is compatible with keeping the contribution from the shear stress term finite. All types of such non-Eulerian inviscid flow are found for the two-dimensional steady axisymmetric vortex.
Keywords : Navier-Stokes equations, non-Eulerian inviscid vortex

1. Introduction

The NS equations for incompressible flow are expressed as

$$\partial_t \boldsymbol{v} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} = \nu\nabla^2\boldsymbol{v} - \frac{\nabla p}{\rho} + f \tag{1.1}$$

with the obvious notations. The first term on the r.h.s. expresses the shear stress due to viscosity. The inviscid fluid is customarily supposed to be described by dropping this Laplacian term. The resultant first order differential equation is called the Euler equation and has been used to understand large scale meteorological phenomena in which the shear stress is negligible as compared to remaining terms in (1.1).

Another way to take the zero viscosity limit is to divide the both sides of (1.1) by ν and then let ν approach zero

$$\lim_{\nu\to 0}\frac{1}{\nu}\left(\partial_t \boldsymbol{v} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} + \frac{\nabla p}{\rho} - f\right) = \lim_{\nu\to 0}\nabla^2\boldsymbol{v}. \tag{1.2}$$

Here each component of the velocity field is assumed to be a function of ν. If (1.2) leads to equations with non-trivial solutions, they will describe the inviscid flow that is controlled by the Laplacian term. We shall call such flows as the non-Eulerian inviscid flows (NEIFs).

(1.2) is the second order differential equation and is expected to lead to a new class of inviscid flow that the Euler equation does not cover. In this paper, one example of the NEIF is presented for a vortex motion of the incompressible fluid, which may be called non-Eulerian inviscid vortex (NEIV).

2. NS equations in cylindrical coordinate

For the later convenience, we here write down the NS equations in the cylindrical coordinate on the slowly rotating frame (rotation of the frame is not essential for our arguments) as

$$\partial_t v_r + v_r \partial_r v_r + \frac{v_\theta}{r}\partial_\theta v_r + v_z \partial_z v_r - \frac{v_\theta^2}{r} = \nu\left(\nabla^2 v_r - \frac{v_r}{r^2} - \frac{2}{r^2}\partial_\theta v_\theta\right) - \frac{1}{\rho}\partial_r p + f_r, \tag{2.1a}$$

$$\partial_t v_\theta + \frac{v_r}{r}\partial_r(r v_\theta) + \frac{v_\theta}{r}\partial_\theta v_\theta + v_z \partial_z v_\theta = \nu\left(\nabla^2 v_\theta - \frac{v_\theta}{r^2} + \frac{2}{r^2}\partial_\theta v_r\right) - \frac{1}{\rho r}\partial_\theta p + f_\theta, \tag{2.1b}$$

$$\partial_t v_z + v_r \partial_r v_z + \frac{v_\theta}{r}\partial_\theta v_z + v_z \partial_z v_z = \nu \nabla^2 v_z - \frac{1}{\rho}\partial_z p + f_z, \tag{2.1c}$$

$$f_r = 2\Omega_z v_\theta + \hat{f}_r,\ f_\theta = -2\Omega_z v_r + \hat{f}_\theta,\ f_z = \hat{f}_z. \tag{2.1d}$$

$\Omega = (0, 0, \Omega_z)$ is the angular frequency vector of the reference frame.　$\hat{f}$ with $\hat{f}_\theta = 0$ is the external force other than the Coriolis force.　The mass conservation is the another condition to be taken into account,

$$\partial_t \rho + \frac{1}{r}\partial_r(r\rho v_r) + \frac{1}{r}\partial_\theta(\rho v_\theta) + \partial_z(\rho v_z) = 0. \tag{2.2}$$

The system is in the Euclidian space with no boundary.

3. ν-expansion and zero viscosity limit in three dimension

Assume that the velocity field and the pressure are functions of ν and are asymptotically expandable as series of ν in the whole Euclidean space with no boundary :

$$X(r, t, \nu) = \sum_{n=0} \nu^n X_n(r, t),\ r \in \mathbb{R}^3. \tag{3.1}$$

(If we consider the time-dependent flow and the regime where the acceleration valances the viscous force, the expansion in $\nu^{1/2}$ will be more pertinent.)　Note that, when the system is steady and has no θ-dependence, the equations (2.1) and (2.2) remain invariant under the transformation $\nu \to -\nu$, $v_r \to -v_r$, $v_\theta \to v_\theta$, $v_z \to -v_z$, $p \to p$.　In this case, therefore, v_r and v_z have odd powers of ν in (3.1), while v_θ and p have even powers.

Physically, it may be more appropriate to adopt, instead of ν, such a dimensionless quantity as the inverse of the Reynolds number for the expansion.　For the present purposes, the expression (3.1) suffices.　We here do not ask an important mathematical question whether the expansion (3.1) always converges everywhere.

Inserting (3.1) to (2.1) and comparing the terms in both sides, we have equations for $\boldsymbol{v}_n(r, t)$.　We here give the equations derived from ν^0, ν^1 and ν^2 terms.

(i). $O(\nu^0)$ equations

These give nothing but the Euler equation :

$$\partial_t \boldsymbol{v}_0 + \boldsymbol{v}_0 \cdot \nabla \boldsymbol{v}_0 + \frac{\nabla p_0}{\rho} - f = 0, \tag{3.2a}$$

$$\partial_t \rho + \nabla(\rho \boldsymbol{v}_0) = 0. \tag{3.2b}$$

We are interested in a field configuration for an incompressible and inviscid fluid that is static and rotationally invariant about the z-axis *for all of* $\boldsymbol{v}_n$. This means that the derivative terms with respect to t and θ are dropped. Furthermore, the velocity field is supposed to have the θ component only in the inviscid limit, i.e., $\boldsymbol{v}_0 = (0, v_{\theta 0}, 0)$ where $v_{\theta 0}$ is a function of r only. In this case, we have noted that v_{rn} and v_{zn} are nonzero for odd n, and $v_{\theta n}$ are nonzero for even n. These conditions reduce (3.2a) to

$$-\frac{{v_{\theta 0}}^2}{r} + \frac{1}{\rho}\partial_r p_0 - 2\Omega_z v_{\theta 0} - \hat{f}_r = 0, \tag{3.2c}$$

$$\frac{1}{\rho}\partial_z p_0 - \hat{f}_z = 0. \tag{3.2d}$$

The mass conservation (3.2b) yields a trivial equation 0 = 0. $O(\nu^1)$ and $O(\nu^2)$ equations are similarly derived.

(ii) $O(\nu^1)$ equations

$$\frac{v_{r1}}{r}\partial_r(r v_{\theta 0}) = \nabla^2 v_{\theta 0} - \frac{v_{\theta 0}}{r^2} - 2\Omega_z v_{r1}, \tag{3.3a}$$

$$\frac{1}{r}\partial_r(r\rho v_{r1}) + \partial_z(\rho v_{z1}) = 0. \tag{3.3b}$$

(iii) $O(\nu^2)$ equations

$$v_{r1}\partial_r v_{r1} - 2\frac{v_{\theta 0} v_{\theta 2}}{r} + \frac{\partial_r p_2}{\rho} = \nabla^2 v_{r1} - \frac{v_{r1}}{r^2} + 2\Omega_z v_{\theta 2}, \tag{3.4a}$$

$$v_{r1}\partial_r v_{z1} + v_{z1}\partial_z v_{z1} + \frac{\partial_z p_2}{\rho} = \nabla^2 v_{z1}. \tag{3.4b}$$

There are six equations for unknown six functions. The dynamics implied by (1.2) requires that $\boldsymbol{v}_0$ and $\boldsymbol{v}_1$ should make a balance. (3.4a) is used to determine $v_{\theta 2}$ from $v_{\theta 0}$, v_{r1}, v_{z1} and p_2. Therefore, the equations relevant for forming NEIF are (3.2c), (3.2d), (3.3a), (3.3b) and (3.4b).

Let us seek solutions in which v_{rn} and $v_{\theta n}$ do not depend on z. Then, from (3.4a), p_2 must be a sum of a function of r and a function of z. On the other hand, (3.4b) implies that the z-dependent part of p_2, if any, is also a function of r. The simplest way to reconcile these situations will be to assume that $\partial_z p_2$ is a constant multiplied by a z-dependent factor common to the remaining terms in (3.4b),

$$\frac{\partial_z p_2}{\rho} = -h(z), \tag{3.5}$$

where $h(z)$ is a function of z to be determined later. In the inviscid limit, v_{r1} and v_{z1} become irrelevant and the flow is described only by $v_{\theta 0}$.

4. Vortex solution

Now, the equations (3.2c), (3.2d), (3.3a), (3.3b) and (3.4b) are viewed to form themselves in three groups. (3.2c) and (3.2d) are used to determine the r and z dependences of p, respectively, when $v_{\theta 0}$ is known. (3.3a) is utilized to solve for $v_{\theta 0}$ when v_{r1} and v_{z1} are known. (3.3b) and (3.4b) can be used to determine v_{r1} and v_{z1}.

Consider first (3.3a), which is rewritten as

$$v_{\theta 0}{}'' + \left(\frac{1}{r} - v_{r1}\right) v_{\theta 0}{}' - \left(\frac{1}{r^2} + \frac{v_{r1}}{r}\right) v_{\theta 0} = 2\Omega_z v_{r1}. \tag{4.1}$$

The prime denotes the derivative with respect to r. It is notable that, even in the absence of boundary, the inviscid flow $v_{\theta 0}$ is affected from the viscous component v_{r1}. The two independent solutions for the homogeneous equation for $v_{\theta 0}$ are $1/r$ and $r^{-1}\int_0^r drre^{\int_a^r dr' v_{r1}(r')}$. The particular solution is $-\Omega_z r$ that expresses the inertial 'motion' of the fluid at rest relative to the rotating frame. The general regular solution is given by

$$v_{\theta 0} = \frac{c}{r}\int_0^r drre^{\int^r dr' v_{r1}(r')} - \Omega_z r. \tag{4.2}$$

In order for $v_{\theta 0}$ given by (4.2) to be finite at infinity, $v_{r1}(\infty)$ must be negative. The second term on the r.h.s. of (4.2) is not essential for our arguments and is disregarded hereafter.

Next consider (3.3b) and (3.4b). Since $v_{\theta 0}$ is assumed to be a function of r only, v_{r1} and $\partial_z v_{z1}$ are functions of r only. The flow is necessarily of the three dimensions. The continuity (3.3b) suggests that

$$v_{z1} = -\frac{z}{r}(rv_{r1})'. \tag{4.3}$$

Thus, $h(z)$ in (3.5) is proportional to z, i.e., $h(z)=c_2 z$. Substituting (4.3) into (3.4b), together with (3.5), yields

$$\frac{1}{r}\left(r\left(\frac{(rv_{r1})'}{r}\right)'\right)' - v_{r1}\left(\frac{(rv_{r1})'}{r}\right)' + \left(\frac{(rv_{r1})'}{r}\right)^2 = c_2. \tag{4.4}$$

Near $r \sim 0$, the solutions of (4.4) for $c_2 \neq 0$ behave as $v_{r1} \propto r$. Thus far two solutions are known

$$v_{r1,\mathrm{B}} = -kr, \tag{4.5a}$$

$$v_{r1,\mathrm{S}} = -kr + 6\left(1 - \exp\left(-kr^2/2\right)\right)/r. \tag{4.5b}$$

The former and the latter are the Burgers vortex (Burgers 1948) and the Sullivan vortex (Sullivan 1984), respectively.

An insight into the possible global behaviours of the solution of (4.4) is gained by rewriting it in terms of a new variable x such that

$$x = -\frac{v_{z1}}{z} = \frac{\left(rv_{r1}\right)'}{r}, \tag{4.6a}$$

or

$$v_{r1} = \frac{1}{r}\int_0^r xrdr \tag{4.6b}$$

as

$$x'' = -x^2 + c_2 + \left(v_{r1} - \frac{1}{r}\right)x'. \tag{4.7}$$

If we reinterpret r as the ‘time’ variable and x as the ‘coordinate’ of a point particle, then this equation expresses the classical one-dimensional motion of the particle with a unit mass in the potential $U(x) = x^3/3 - c_2 x$ under the effect of some non-conservative force given by the last term on the r.h.s. of (4.7). An example of the form of the potential U is depicted in Fig. 1 for $c_2=1$.

Multiply the both sides of (4.7) by x' and rewrite the resultant equation to obtain

$$\frac{d}{dr}\left(\frac{1}{2}x'^2 + U(x)\right) = \left(v_{r1} - \frac{1}{r}\right)x'^2. \tag{4.8}$$

This equation means that the temporal variation rate of the particle’s ‘total energy’ is governed by the non-conservative force involved on l.h.s. If the l.h.s. of (4.6) is zero, then the energy is conserved. This is achieved by resting the particle at the one of the extrema of the potential. Discarding the positive value by the reason already mentioned, the acceptable solution is

$$x = -1 \text{ or } v_{r1} = -\frac{r}{2}. \tag{4.9}$$

This is shown as the point A in Fig. 1, which is nothing but the Burgers vortex solution with k=1/2 with the ‘energy’ equal to 2/3. We require any physical solution to asymptotically approach the point A.

The particle at rest generally begins to roll down the potential slope. The behaviour of $x(r)$ and the corresponding increase of the speed of the real flow near $r = 0$ will be written as

$$x(r) \approx x(0) + a_2 r^2, \tag{4.10a}$$

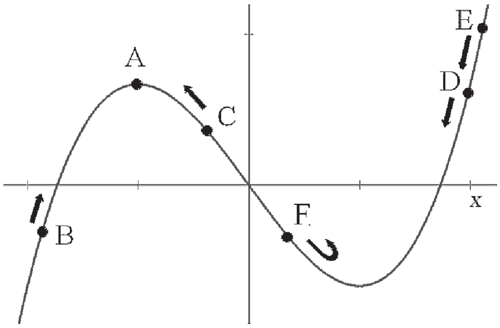

Fig. 1　Potential $U = x^3/3 - c_2 x$ with c_2=1. The maximum and minimum points are x=−1 and 1, respectively. This graph shows why $c_2 > 0$ is necessary for physical solutions to exist. For alphabetical letters and arrows, see the text.

$$v_{r1}(r) \approx \frac{x(0)}{2} r + \frac{a_2}{4} r^3, \tag{4.10b}$$

where $x(0)$ and a_2 are given by the initial conditions. The particle loses the initial energy due to the dissipative term $-x'^2/r$. Even in the case v_{r1} temporarily acquires positive values, on approaching the point A, the functional form of v_{r1} should approach $-r/2$, thereby the non-conservative force eventually turns totally dissipative. By appropriately choosing the initial position, the particle will approach A at $r = \infty$. One of such motions corresponds to the Sullivan's vortex, for which $x_{\text{Sullivan}}(0) = 2$ and $a_2 = -3/2$, and is designated by D in Fig. 1. Its initial energy is 2/3, being equal to the final state's energy.

When the particle's initial position is B in Fig. 1, then it can climb up the slope if the acceleration $x''(0)$ has a positive sign at B. By appropriately choosing the acceleration at B, the particle can be in the stationary state A at $r = \infty$. The similar thing holds for C, E and F : if the particle has an appropriate negative acceleration at these points, then it can climb up the slope and get stationary at A.

Interestingly, the particle at the initial point C or F can have a positive initial acceleration, i.e., $x''(0) > 0$, to reach the point A. It moves down and then up the slope beyond the minimum, stops at a certain point, turns the direction of motion and climbs down and then up toward the point A.

These peculiar characteristics of the points C and F are due to the existence of v_{r1} in the non-conservative force of our fictitious classical dynamics.

v_{r1} is easily determined by directly solving (4.4). $r = 0$ is a singularity of (4.4), so that the numerical integrations were started from r_0 near $r = 0$. We set $r_0 = 0.02$. The results are shown in Fig. 2 for the six types of initial conditions mentioned above. Since c_2 in (4.4) is fixed to unity, all solutions asymptotically approach $\sim r/2$. The profiles labeled A, B and C are almost indistinguishable each

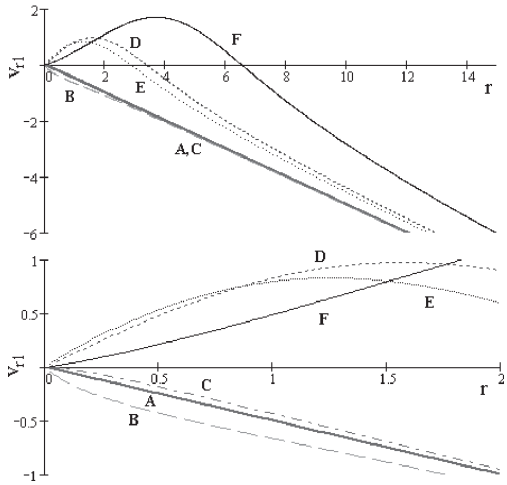

Fig. 2　Six types of the r-dependence of v_{r1}.　The characters A~F correspond to those in Fig. 1.　The initial conditions are A (thick solid curve) : $x(r_0)=-1$, $x'(r_0)=0$; B (long-dashed curve) : $x(r_0)=-2$, $x'(r_0)=14.339$; C (dash-dotted curve) : $x(r_0)=-0.5$, $x'(r_0)=-6.901$; D (short-dashed curve) : $x(r_0)=2$, $x'(r_0)=0$; E (dotted curve) : $x(r_0)=3$, $x'(r_0)=-13.008$; F (thin solid curve) : $x(r_0)=0.4$, $x'(r_0)=7.723$.　r_0 is taken to be 0.02.　The enlarged one in $0 < r < 2$ is shown in lower panel.

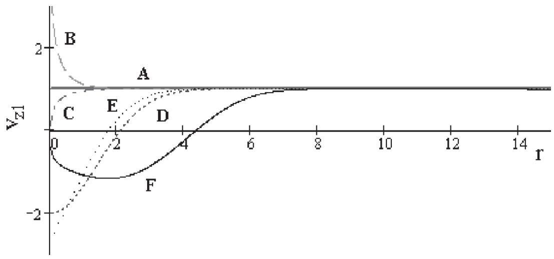

Fig. 3　v_{z1} at $z = 1$.　The line types and the meanings of the characters A～F are same as in Fig. 2.

other in large scales and exhibit inward flows in all space.　Those labeled D, E and F exhibit the so called two-cell structure : the direction of the flow is outward near the symmetry axis and inward in outer region.

v_{z1} is determined from (4.3).　The six profiles corresponding to those in Fig. 2 are shown in Fig. 3.　For all of the flows, $v_{z1}(\infty)=1$.　This figure shows that the flows A, B and C are upward.　In the Burgers vortex A, $\partial_r v_{z1} \equiv 0$.　On the other hand, on approaching the symmetry axis, v_{z1} looks to increase (decrease) for the flow B (C).　v_{z1} changes the sign at some radius for the flows D, E and F, so that their two-cell structures are obvious.

$v_{\theta 0}$ determined by (4.2) with $\Omega_z = 0$ are shown in Fig. 4 for the flows A, C, D, E and F.　Far away from the maximum point, $v_{\theta 0}$ decreases as $1/r$.　Near the symmetry axis, $v_{\theta 0}$ is proportional to r for

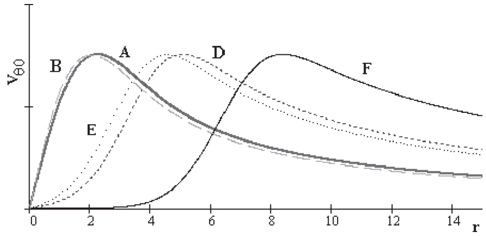

Fig. 4　$v_{\theta 0}$.　The line types and the characters A〜F are of the same meanings as in Fig. 2.　The normalization is arbitrary.　The profile C almost coincides with A and is not shown here.

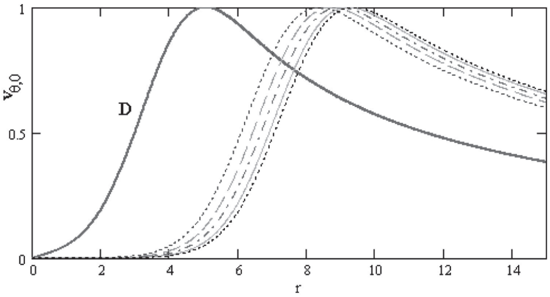

Fig. 5　r_0 dependences of $v_{\theta 0}$ for the numerical solution of the eye-type.　Left-most dotted curve : r_0=0.02, long-dashed curve : 0.01, dash-dotted curve : 0.005, thin solid curve : 0.002, right-most dotted curve : 0.001.　The one marked by 'D' is the Sullivan solution.　The numerical solution of eye-type shifts rightward for $r_0 \to 0$.　All are normalized to one.

all these flows because $x(r_0) \neq 0$.　However, the rates of the subsequent rise are different : $\partial_r v_{\theta 0}$ increases relatively very rapidly in F.　In this flow, $v_{\theta 0}$ diminishes more rapidly than r^1 (probably than any power of r in the limit $r_0 \to 0$) near $r = 0$, thereby forming a clear eye and eye-wall.

By taking the limit $r_0 \to 0$ numerically, we observe (but not shown here) that the solutions B and C approach A rapidly.　Therefore, the solutions B and C are expected to converge to A, i.e., the Burgers solution.　In the same limit, the solution E approaches D slowly.　The flow E is anticipated to converge to D, the Sullivan solution.　The two-cell structured solution F also seems to have $r_0 \to 0$ limit, as is depicted in Fig. 5 for $v_{\theta 0}$.　However, the direction of the convergence is such that the solution gets far apart from D, the Sullivan solution.　Thus we conclude that the type F is the new solution.

The orbit C of the fictitious particle started from a point $x(r_0)$ such that $-1 < x(r_0) < 1$ with a negative acceleration.　Importantly, if it were given an appropriately adjusted positive acceleration, the particle first moved down the slope in Fig. 1, then changed the direction and went up until it reached A.　This orbit looks like that of F but is distinct since the initial position is different.

Similarly, the particle started from the point F with an appropriately tuned negative acceleration can reach A at $r=\infty$. In the limit $r_0 \to 0$, such an orbit seems to coincide with the orbit A, the Burgers solution.

Thus we are left with three types of flows : the Burgers vortex, A, the Sullivan vortex, D, and, so as to say, the eye-type vortex, F. These are characterized by the value of $x(0)=(rv_{r1})'/r\,|_{r=0}$: $x_A(0)=-1$, $x_D(0)=2$ and $-1<x_F(0)<1$, where the subscript specifies the flow-type. Probably, solutions with any other initial position will converge on one of these three types in the limit $r_0 \to 0$.

The radial direction of the flow of the eye-type vortices changes near the middle point of the eye-wall. In the outer and inner regions, the flow directs inward and outward, respectively. The vertical direction of the flow also changes from downward to upward with r at the inner foot of the eye-wall.

5. Summary and comments

We presented one example of the NEIV for a new two-dimensional vortex as the solution of the Navier-Stokes equation. The configuration of the velocity field is governed not only by the advection and the pressure gradient but also by the shear term that is absent in the Euler equation.

The velocity field is assumed to be expanded as an asymptotic series of ν. Then, matching the coefficients of the same power of ν, a set of differential equations for the expansion coefficients were derived. It is interesting that the 0th- and 1st-order coefficients form a closed set of equations when the zero-viscosity limit of the field leaves only the azimuthal component finite. Before the zero viscosity limit is taken, the flow is of a three dimensional and the singularity observed in the two dimensional flow is avoided (Takahashi 2013).

In the zero-viscosity limit, the radial and vertical components of the flow vanish. However, the first order coefficient of the radial and vertical components remains finite. In particular, the radial component is directly related to the profile of the final azimuthal component. Such a 'Cheshire cat' effect is possible in case the flow is of the multicomponent.

In gaining the perspective on the nature of the solutions, it was helpful to translate the fluid dynamical equation to the equation of motion of a point particle in a potential deduced from the NS equations and the continuity equation. This method will enjoy finding wide application for solving the NS equations.

In the new solution, the azimuthal flow is structured by an inner eye, an eye-wall and a decaying tail in outer region, which are reminiscent to those of typhoon. The scaling arguments indicate that smaller ν corresponds to larger Reynolds number. Probably, taking the inviscid limit is a mathematically idealized procedure adaptable for describing the typhoon's growth and strengthening.

The ν-expansion method may in principle be applicable for small but finite ν (see, e.g., Sammartino and Caflisch 1998). In that case, νv_{r1} and νv_{z1} are the approximations of the radial and the vertical component of the flow that enable us to approximately reconstruct the three-dimensional flow. Whether the asymptotic expansion in ν always converges remains as an open question.

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