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Variational Principle for Eulerian Dynamics of Incompressible Viscous Fluid and A New 'Eddy' Viscosity Model

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Abstract : A new Eulerian variational principle that derives the Navier-Stokes equation for incompressible fluid is presented. The Lagrangian is constructed in terms of a field expressed by a two by two complex matrix so as for the stationary action principle to give the correct fluid dynamics. When the matrix field is traceless and can be decomposed to a vector that is identified as the velocity field, the stationary action principle without any additional constraints yields the Navier-Stokes equation. Next, a complex scalar field as the centre of $GL(2,C)$ is introduced into the matrix field. Then, two kinds of extensions of Lagrangian are considered. In the first, the interaction involves terms up to the second order in the field. When the Lagrangian is real and the fields are limited to the real space, the velocity obeys the ordinary Navier-Stokes equation, and the scalar behaves as a diffusive entity. In the second extension, the interaction involves the third power of the field. In the real space of the component fields, the field equation reduces to a system similar to the eddy-viscosity equations with the scalar and the vector being interpreted as the eddy viscosity and the mean velocity, respectively. When applied to turbulent channel and pipe flows, the model of the second extension satisfactorily reproduces the mean velocity profiles observed in experiments over the viscous sublayer and the so called 'logarithmic' region.

Key words : Navier-Stokes equations ; variational principle ; eddy-viscosity model ; turbulence

1. Introduction

The mathematical problem of how the Navier-Stokes equation is derived by the variational principle in classical mechanics has been challenged in many works. The Euler equation for non-dissipative fluid is known in fact to be derived by the variational principle. One way to see this is to consider the Lagrangian path $g(t, g_0)$ of a fluid element, where g_0 is the initial position of the element. The velocity and the acceleration are given by

$$u(t, g) = \frac{\partial}{\partial t} g,$$

$$\frac{d}{dt}u(t, g) = \frac{\partial}{\partial t}u + u \cdot \nabla u.$$

Therefore, by adopting the Lagrangian density of the form

$$\mathrm{L} = \frac{\rho}{2}\dot{g}^2 + g \cdot \nabla p$$

together with the principle of stationary action

$$\delta \int \mathrm{L} dx dt = 0$$

leads to the Euler equation (see, e.g., Yasue 1983). The velocity term in L has a clear physical meaning of the kinetic energy density also constitutes the norm for the Liyapunov stability criterion. For the Hamiltonian formalism, see, e.g., Salmon (1988).

Construction of the action principle for the Eulerian field theory is also possible. This is conveniently achieved by introducing the notion of time-independence of the initial position of each Lagrangian path into the action by employing the Lagrange multipliers (Lin 1963 ; Mittag et al. 1979). Construction of the Eulerian action in view of gauge principle has also been proposed by Kambe (2007).

The presence of the viscosity Δu term changes the situation. This may be intuitively understood by noting that the viscosity term in the N-S equation is linear in u. Therefore, as long as the geodesic g is regarded as the independent variable subjected to variation, one cannot construct a nontrivial term that is bilinear in g.

The way to correctly produce the viscous term through the action principle is to regard the dissipation as a random process, thereby reformulating the variational principle into the stochastic one. In terminology of the probability theory, the idea is based on that the generator of the semigroup of the Markov process is given by the Laplacian Δ that has the counterpart of the viscous force in the N-S equation. This approach was originally addressed by Inoue and Funaki (1979), Yasue (1981), Nakagomi et al. (1981), whose mathematical foundations including the verification of the existence of the stochastic least action have been provided by many authors. For recent works, see, e.g., Cipriano and Cruzeiro (2007), Constantin and Iyer (2008), Eyink (2010) and references cited therein. The velocity in stochastic formalism is therefore regarded to represent the dissipation velocity.

Since the N-S equation for the velocity field is derived within classical deterministic dynamics, it is natural to ask if a field theory that is defined by deterministic action principle is possible. One approach along this line was surveyed by Fukagawa and Fujitani (2012) which succinctly preserves the notion of deterministic path of particle in classical mechanics. Employing the Lagrange multipli-

ers to guarantee the energy conservations in closed system, they showed that the N-S equation can be derived by the minimization of the effective action that incorporates the constraints by the Lagrange multiplier method. In their model, however, the N-S equation is not at the total derivative of the effective action since one of the constrains is necessarily non-holonomic. Here, a constraint is called non-holonomic when it is an equation of motion of some physical quantity that is not derivable from a total derivative of action.

The author (Takahashi 2016) previously proposed a mean field model of turbulence based on the variational principle with a constraint and showed that the mean velocity calculated in the model is entirely consistent with some experiments on turbulent flow. The constraint adopted there was also non-holonomic.

It seems difficult to incorporate the dissipative process into a framework of deterministic field theory based on the canonical action principle unless the meaning of the Hamiltonian is altered (Salmon 1988). Salmon gave a following example. Let us take the heat conduction equation : $\dot{T}-\lambda\Delta T=0$ with a positive constant λ. The action may be written as $\int \alpha(\dot{T}-\lambda\Delta T)d\boldsymbol{r}dt$, where the Lagrange multiplier α is an auxiliary field that obeys a cohesive equation $\dot{\alpha}=-\lambda\Delta\alpha$. This leads to a 'Hamiltonian' $-\int \lambda\nabla\alpha\cdot\nabla T d\boldsymbol{r}$. The physical meaning of the auxiliary field α is unclear, although T and α emerge symmetrically.

To the author's knowledge, no Eulerian field theory of viscous fluid based on the deterministic action principle has been found. The purpose of the present paper is to propose a method of the deterministic stationary action principle that leads to the N-S equation for incompressible fluid. The procedure is as follows. We extend the velocity field to a complex valued one and write down the action. Minimization of the action is done in the complex space of the velocity field, followed by restricting the velocity field be real. Then, the desired equation results. In other words, we are going to pursue the course opposite to the standard classical mechanics.

Our method will be conceptually similar to the Salmon's example mentioned above, with the imaginary part of the velocity playing a role of the auxiliary field. The components of the velocity are assembled to a field of a single 2 by 2 complex matrix in terms of which the action is expressed without using any Lagrange multipliers. Although the stationary action principle leads to the correct Hamilton's equations, our 'action' turns out not to have the standard form of the kinetic energy minus the potential energy. Consequently, we will have an unusual conserved 'Hamiltonian'. Therefore, in this paper, we shall call our action 'pseudo-action' (, while, for brevity, the familiar terms 'Lagrangian' and 'Hamiltonian' will be used).

The stationary action principle furnishes a closed and consistent set of equations with required symmetries. This property can be preserved after any modification of the pseudo-action. We will then proceed to extend the matrix field to incorporate an additional scalar field. In this way, we will finally have a model of field theory that is akin to the so-called one equation eddy viscosity model (OEEVM, Spalart and Allmaras 1992 ; 1994) in the Eulerian description of motions. An application of the model to turbulent flows will be presented.

This paper is organized as follows. In the next section, we show how the pseudo-action for fluid dynamics is constructed in terms of the complex matrix field that describes the velocity. In sec.3, the matrix field is extended to incorporate a scalar field. The resultant equations of motion turn out to work as a OEEVM and their outcomes are compared with experimental results. Final section is devoted to summary and some remarks.

2. Pseudo-action for incompressible fluid and the N-S equation

2.1 *Construction of pseudo-action and variational principle*

In this section, we elaborate our method by considering the pure N-S equation. The N-S equation consists of the Lagrangian derivative (which is also called material derivative, convective derivative and so on) terms, the dissipation term and the force term. We write the corresponding contributions to the pseudo-action A_{NS} as A_{Ld}, A_{dis} and A_{f} :

$$A_{\mathrm{NS}}=A_{\mathrm{Ld}}+A_{\mathrm{dis}}+A_{\mathrm{f}}. \tag{2.1.1}$$

Let Φ be a complex 2×2 matrix which is equivalent to a complex velocity vector $\boldsymbol{u}$. In order to match the degree of freedom, Φ is restricted to be traceless. Suppose the following expression for A_{Ld}

$$A_{\mathrm{Ld}}=i\mathrm{Tr}\int\left(a\Phi^{\dagger}\dot{\Phi}+b\Phi^{\dagger}\sigma_i\Phi^{\dagger}\partial_i\Phi-b\partial_i\Phi^{\dagger}\Phi\sigma_i\Phi+\mu_a\Phi^{\dagger}-\mu_a\Phi\right)d\boldsymbol{r}dt, \tag{2.1.2}$$

where $\Phi^{\dagger}$ is the hermitian conjugate of Φ. The dot on Φ stands for a derivative with respect to time. a and b are constants to be determined by comparing the result with the Euler equations. We impose a condition that, in analogy with the classical mechanics, the inertial time-derivative term gives a pure real contribution to A_{Ld} modulo surface integration. This means a is real. Summations for repeated indices are implied. $\sigma_i, i=1(=x), 2(=y), 3(=z)$ are the Pauli's matrices :

$$\sigma_i=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{2.1.3}$$

These satisfy the following commutation and anticommutation relations

$$[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k, \{\sigma_i, \sigma_j\} = 2\delta_{ij}. \tag{2.1.4}$$

μ_a is a Lagrange multiplier introduced to constrain Φ to be traceless.

The variation of A_{Ld} under a small variation of $\Phi^\dagger$ is given in the Cartesian coordinate system by

$$\begin{aligned}\delta A_{Ld} &= i\mathrm{Tr}\int \delta\Phi^\dagger\left[a\dot{\Phi}+b\sigma_i\Phi^\dagger\partial_i\Phi+b\partial_i\Phi\Phi^\dagger\sigma_i+b\partial_i(\Phi\sigma_i\Phi)+\mu_a\right]d\boldsymbol{r}dt \\ &= i\mathrm{Tr}\int \delta\Phi^\dagger\mathbf{M}d\boldsymbol{r}dt.\end{aligned} \tag{2.1.5}$$

M stands for the matrix in the brackets in (2.1.5). a and b are real constant. We determine a and b so as for **M** to reduce to the Lagrangian derivative terms in the N-S equation. For this purpose, we decompose Φ as

$$\Phi(\boldsymbol{u}) = u_i\sigma_i \equiv \boldsymbol{u}\cdot\boldsymbol{\sigma}, \quad \det\Phi(\boldsymbol{u}) = -\boldsymbol{u}^2. \tag{2.1.6}$$

$\boldsymbol{u}$ is a complex vector field, which will finally be set to be real.

The equation of motion for Φ should be equivalent to the equations of motion for u_i, which are given by the coefficient when **M** is decomposed in σ_i. Multiply **M** by $\boldsymbol{\sigma}$ and take a trace. After some manipulations with uses of (2.1.4), we have

$$\mathrm{Tr}\sigma_i\mathbf{M} = 2a\dot{u}_i + 8b\boldsymbol{u}\cdot\nabla u_i. \tag{2.1.7}$$

Thus, the correct Lagrangian derivative results by choosing $a=1/2$ and $b=1/8$. Thus, the form of A_{Ld} reads

$$A_{Ld} = i\mathrm{Tr}\int\left(\frac{1}{2}\Phi^\dagger\dot{\Phi}+\frac{1}{8}\Phi^*\sigma_i\Phi^*\partial_i\Phi-\frac{1}{8}\partial_i\Phi^\dagger\Phi\sigma_i\Phi\right)d\boldsymbol{r}dt. \tag{2.1.8}$$

Similarly, the trace of **M** gives a scalar part of the equation

$$\mathrm{Tr}\mathbf{M} = 2\mu_a. \tag{2.1.9}$$

For the dissipation term, let us try the form

$$A_{dis} = \frac{ic_0}{4}\int \mathrm{Tr}(\partial_i\Phi^\dagger\partial_i\Phi^\dagger-\partial_i\Phi\partial_i\Phi)d\boldsymbol{r}dt. \tag{2.1.10}$$

The variation of A_{dis} yields

$$\delta A_{dis} = i\int \mathrm{Tr}\delta\Phi^\dagger\mathbf{D}d\boldsymbol{r}dt, \quad \mathbf{D} \equiv -\frac{c_0}{2}\nabla^2\Phi^\dagger. \tag{2.1.11}$$

This expression gives

$$\mathrm{Tr}\sigma_i\mathbf{D} = -c_0\nabla^2u_i, \tag{2.1.12}$$

$$\mathrm{Tr}\mathbf{D} = 0. \tag{2.1.13}$$

Requiring (2.1.12) to coincide with the dissipation term in the N-S equation, c_0 turns out to be equal to the kinematic viscosity ν.

$\tilde{\boldsymbol{f}} \equiv -\rho^{-1}\nabla\rho + \boldsymbol{f}$, the force term in the N-S equation, is a real vector and includes the body force and the pressure gradient. It will be derived by adopting the from

$$A_{\mathrm{f}} = \frac{ie}{2}\int \mathrm{Tr}(-\Phi^{\dagger}\mathbf{F} + \mathbf{F}^{\dagger}\Phi)d\boldsymbol{r}dt, \quad \mathbf{F} \equiv \tilde{\boldsymbol{f}}\cdot\boldsymbol{\sigma} = \mathbf{F}^{\dagger} \tag{2.1.14}$$

with a constant e. In fact, the variation of A_{f}

$$\delta A_{\mathrm{f}} = -\frac{ie}{2}\int \delta\Phi^{\dagger}\mathbf{F}d\boldsymbol{r}dt, \tag{2.1.15}$$

yields for the traces

$$-\frac{e}{2}\mathrm{Tr}\sigma_i\mathbf{F} = -e\tilde{f}_i, \tag{2.1.16}$$

with $\mathrm{Tr}\mathbf{F} = 0$. (2.1.16) results in $e = 1$. Putting A_{Ld}, A_{dis} and A_{f} all together, we finally have the total pseudo-action A_{NS} and the Lagrangian L_{NS} of the forms

$$A_{\mathrm{NS}} = \int L_{\mathrm{NS}}dt \equiv \int \mathrm{L}_{\mathrm{NS}}d\boldsymbol{r}dt, \tag{2.1.17a}$$

$$\mathrm{L}_{\mathrm{NS}} = i\mathrm{Tr}\Big(\frac{1}{2}\Phi^{\dagger}\dot{\Phi} + \frac{1}{8}\Phi^{\dagger}\boldsymbol{\sigma}\Phi^{\dagger}\cdot\nabla\Phi - \frac{1}{8}\nabla\Phi^{\dagger}\cdot\Phi\boldsymbol{\sigma}\Phi + \frac{c_0}{4}(\nabla\Phi^{\dagger})^2 - \frac{c_0}{4}(\nabla\Phi)^2 - \frac{1}{2}\Phi^{\dagger}\mathbf{F} + \frac{1}{2}\mathbf{F}\Phi\Big), \tag{2.1.17b}$$

together with $\mu_{\mathrm{a}} = 0$. Variational principle $\delta A_{\mathrm{NS}}/\delta\Phi^{\dagger} = 0$ yields

$$\dot{\Phi} + \frac{1}{4}\boldsymbol{\sigma}\Phi^{\dagger}\cdot\nabla\Phi + \frac{1}{4}\nabla\Phi\cdot\Phi^{\dagger}\boldsymbol{\sigma} + \frac{1}{4}\nabla\cdot(\Phi\boldsymbol{\sigma}\Phi) = c_0\nabla^2\Phi^{\dagger} + \mathbf{F}. \tag{2.1.18}$$

Since A_{NS} is real, L_{NS} must constitutes of odd power of $\mathrm{Im}\boldsymbol{u}$. Therefore, variation of the real parts of $\boldsymbol{u}$ yields the equations of motion that constitutes of odd power of the imaginary parts. Those equations thus always have $\mathrm{Im}\boldsymbol{u} = 0$ and $\Phi^{\dagger} = \Phi$ as a solution. Then, the sum of the second, third and fourth terms on l.h.s. of (2.1.18) is written for real $\boldsymbol{u}$ as

$$\frac{1}{4}\boldsymbol{\sigma}\Phi\cdot\nabla\Phi + \frac{1}{4}\nabla\Phi\cdot\Phi\boldsymbol{\sigma} + \frac{1}{4}\nabla\cdot(\Phi\boldsymbol{\sigma}\Phi) = \frac{1}{4}\{\boldsymbol{\sigma}, \Phi\}\cdot\nabla\Phi + \frac{1}{4}\nabla\Phi\cdot\{\Phi, \boldsymbol{\sigma}\}$$

$$= \boldsymbol{u}\cdot\nabla\Phi$$

so that (2.1.18) takes the form

$$\dot{\Phi} + \boldsymbol{u}\cdot\nabla\Phi = c_0\nabla^2\Phi + \mathbf{F}. \tag{2.1.18'}$$

The vector component of (2.1.18') is extracted by taking trace after multiplying (2.1.18') by σ_i. Scalar component is given by taking trace of (2.1.18), which identically vanishes. Finally, we obtain the

N-S equations

$$\dot{u}_i + \boldsymbol{u}\cdot\nabla u_i = c_0\nabla^2 u_i - \frac{\partial_i p}{\rho} + f_i. \tag{2.1.19}$$

Consequently, it was shown that the N-S equations are derived by requiring the *total* derivative of A_{NS} be stationary under unconstrained small variations of the field. The imaginary parts of the velocity field play the role analogous to a Lagrange multiplier.

2.2 *Symmetries and invariants*

The Lagrangian can be invariant under some global transformation of $\Phi:\Phi(\boldsymbol{u}(r))\to\Phi'(\boldsymbol{u}'(\boldsymbol{r}'))$. If **F** is uniform, temporal and spatial translations are symmetry and Noether's theorem states that the energy-momentum tensor T defined by

$$\mathrm{T}_{0\nu} = -\delta_{0\nu}\mathrm{L}_{\mathrm{NS}} + \mathrm{Tr}\Pi\partial_\nu\Phi$$

is conserved. $\Pi = i\Phi^{\dagger T}/2$ is the canonically conjugate momentum of Φ. The conserved 'energy' for the present action is

$$H = i\int \mathrm{Tr}\Big(-\frac{1}{8}\Phi^\dagger\boldsymbol{\sigma}\Phi^\dagger\cdot\nabla\Phi + \frac{1}{8}\nabla\Phi^\dagger\cdot\Phi\boldsymbol{\sigma}\Phi - \frac{\nu}{2}(\nabla\Phi^\dagger)^2 + \frac{\nu}{2}(\nabla\Phi)^2 + \frac{1}{2}(\Phi^\dagger\mathbf{F} - \mathbf{F}^\dagger\Phi)\Big)d\boldsymbol{r}, \tag{2.2.1}$$

By symmetrizing the time-derivative term in L_{NS}, the 'three momentum' is given by

$$\boldsymbol{P} = \frac{i}{4}\int \mathrm{Tr}(\Phi^\dagger\nabla\Phi - \nabla\Phi^\dagger\Phi)d\boldsymbol{r}. \tag{2.2.1}$$

H and $\boldsymbol{P}$ identically vanish for real $\boldsymbol{u}$ because, in this case, $\Phi^\dagger = \Phi$. However, it is easily checked that the above H generates the Hamilton's equations

$$\dot{\Phi} = \frac{\delta H}{\delta\Pi}, \quad \dot{\Pi} = -\frac{\delta H}{\delta\Phi}, \tag{2.2.2}$$

and in this sense H is the Hamiltonian.

The rotation of angle θ of the coordinate system about the axis of the direction of unit vector $\boldsymbol{n}$, the coordinate transforms as $\boldsymbol{r}'^T = \mathbf{R}(\theta,\boldsymbol{n})\boldsymbol{r}^T$ with $\mathbf{R}(\theta,\boldsymbol{n})$ being a 3×3 matrix representation of the group $SO(3)$. We write such transformation symbolically as $\boldsymbol{r}' = \mathbf{R}\boldsymbol{r}$. Then, transformation of any vector $\boldsymbol{\nu}$ is written in a same way :

$$\boldsymbol{\nu}'(\boldsymbol{r}') = \mathbf{R}\boldsymbol{\nu}(\boldsymbol{r}'). \tag{2.2.3}$$

There is a unitary operator

$$U_\sigma(\theta) = e^{-i\theta\boldsymbol{n}\cdot\boldsymbol{\sigma}/2} \tag{2.2.4}$$

that induces the transformation of $\boldsymbol{\sigma}$ as a vector

$$\boldsymbol{\sigma}' = U_\sigma(\theta)\boldsymbol{\sigma}U_\sigma(\theta)^{-1} = \mathbf{R}\boldsymbol{\sigma}. \tag{2.2.5}$$

Therefore, Φ is invariant under the transformation

$$\Phi(\boldsymbol{u}) \to \Phi'(\boldsymbol{u}') = U_\sigma \Phi(\mathbf{R}\boldsymbol{u})U_\sigma^{-1} = \Phi(\boldsymbol{u}). \tag{2.2.6}$$

Owing to this property of Φ, if $\boldsymbol{f}$ is a vector, $\mathbf{F}$ and the Lagrangian too are invariant under the rotation. Nullity of the total angular momentum in real space also follows. These are the reason we called A_{NS} the pseudo-action.

That H is conserved for any force $\mathbf{F}$ as long as Φ is a solution of the equation of motion holds for arbitrary complex $\mathbf{F}$. Let us consider to add to $\mathbf{F}$ in (2.2.1a) an infinitesimal imaginary component $\mathbf{F}'$. Due to the Helmholtz's theorem, $\mathbf{F}'$ may be written as

$$\mathbf{F}' = i\boldsymbol{f}'\cdot\boldsymbol{\sigma} = i(\nabla k + \nabla\times\boldsymbol{h})\cdot\sigma, \tag{2.2.7}$$

where k and $\boldsymbol{h}$ are time-independent arbitrary infinitesimal real one- and three-components functions, respectively. Suppose $\boldsymbol{u}$ is real. Adding an antihermitian matrix $\mathbf{F}'$ to $\mathbf{F}$ in the equation of motion will result in emerging an infinitesimal imaginary part $\boldsymbol{u}'$ in the velocity. Let us replace the corresponding Φ as

$$\Phi \to \Phi + \Phi' = \boldsymbol{u}\cdot\boldsymbol{\sigma} + i\boldsymbol{u}'\cdot\boldsymbol{\sigma}. \tag{2.2.8}$$

$\Phi' = i\boldsymbol{u}'\cdot\boldsymbol{\sigma}$ is antihermitian. The change H' in the Hamiltonian (2.2.1) in the lowest order of $\mathbf{F}'$ is symbolically expressed by

$$H' = \mathrm{Tr}\int\left[\frac{i}{2}\Phi^\dagger\dot{\Phi}' + \frac{i}{2}\Phi'^\dagger\dot{\Phi} - (\delta L_{\mathrm{NS}}/\delta\Phi^\dagger)\Phi'^\dagger - (\delta L_{\mathrm{NS}}/\delta\Phi)\Phi' + (\delta L_{\mathrm{NS}}/\delta\mathbf{F}^\dagger)\mathbf{F}' - (\delta L_{\mathrm{NS}}/\delta\mathbf{F})\mathbf{F}'\right]d\boldsymbol{r},$$

where L_{NS} has been treated as a functional of independent fields Φ and $\mathbf{F}$. $\mathbf{F}'^\dagger = -\mathbf{F}'$ has also been used. The third and fourth terms vanish because of the equations of motion for Φ. In addition, noting that $\delta L_{\mathrm{NS}}/\delta\mathbf{F}^\dagger = i\Phi/2 = -\delta L_{\mathrm{NS}}/\delta\mathbf{F}$, H' is rewritten as

$$\begin{aligned} H' &= i\mathrm{Tr}\int\left(\frac{1}{2}\Phi\dot{\Phi}' - \frac{1}{2}\dot{\Phi}\Phi' + \Phi\mathbf{F}'\right)d\boldsymbol{r} \\ &= \int(-\boldsymbol{u}\cdot\dot{\boldsymbol{u}}' + \dot{\boldsymbol{u}}\cdot\boldsymbol{u}' - 2\boldsymbol{u}\cdot\nabla k - 2\boldsymbol{u}\cdot\nabla\times\boldsymbol{h})d\boldsymbol{r}. \end{aligned} \tag{2.2.9}$$

The integration region is an infinite cylinder with a cross section S of very large radius R. H' is a constant of motion when small perturbation exists.

Let us consider the case of $k=0$. Suppose that the flow is two-dimensional, i.e., $\boldsymbol{u} = (u_x, u_y, 0)$. Choose the form $\beta = (0, 0, \zeta(x, y))$ in $-\ell \le z \le \ell$, where $\zeta = \zeta_0 = \text{constant} \neq 0$ within radius=

$r \leq R_1$ and $\zeta = 0$ at $r = R > R_1$ with $R - R_1 \ll R$. ζ monotonically decreases in $R_1 \leq r \leq R$. In terminology of mathematics, the interior of a circle C with $r = R$ is the support of ζ. For this configuration of $\boldsymbol{h}$, $\boldsymbol{f}' = \nabla \times \boldsymbol{h}$, and $\boldsymbol{u}'$ too, are non-vanishing only in the narrow region $R_1 \leq r \leq R$. Therefore, the first two $\boldsymbol{u}'$-dependent terms in (2.2.9) can be neglected. Then, H' per unit length along the z-axis is written as

$$H'/2\ell = -2\int_S \boldsymbol{u}\cdot\nabla\times\boldsymbol{h}dS$$

$$= 2\int_S \nabla\cdot(\boldsymbol{u}\times\boldsymbol{h})dS - 2\int_S (\nabla\times\boldsymbol{u})\cdot\boldsymbol{h}dS.$$

The integrations on r.h.s. are assumed to exist. By virtue of Gauss' theorem, the first term on r.h.s. vanishes :

$$2\int_S \nabla\cdot(\boldsymbol{u}\times\boldsymbol{h})dS = 2\int_C (\boldsymbol{u}\times\boldsymbol{h})\cdot d\boldsymbol{l} = 0$$

because $\zeta = 0$ on C. Here, $\boldsymbol{dl}$ is an oriented line element on C. In the second term, $\boldsymbol{h}dS$ can be replaced by $\zeta_0 d\boldsymbol{S}$ where $d\boldsymbol{S}$ is an area element directed to the positive z-direction. Applying Stokes' theorem, we have

$$H'/2\ell \approx -2\zeta_0 \oint \boldsymbol{u}\cdot d\boldsymbol{l}. \tag{2.2.10}$$

By letting $R, R_1 \to \infty$ with $R - R_1 \ll R$, the approximate equality in (2.2.10) becomes the equality. The r.h.s. of (2.2.10) is conserved for any constant ζ_0, so that we have the conservation of circulation at infinity

$$\frac{d}{dt}\oint \boldsymbol{u}\cdot d\boldsymbol{l} = 0. \tag{2.2.11}$$

(2.2.11) corresponds to Kelvin's theorem for inviscid flow. The derivations of the fluid equation or the exact Kelvin's theorem for inviscid flow within the Eulerian stationary action principle have been presented by many authors (Lin 1965, Van Saarloos 1981, Salmon 1988). Their actions are of the standard classical mechanics : the Lagrangian is constructed from kinetic energy minus thermodynamical internal energy supplemented by several constraints with uses of Lagrange multipliers. For instance, the so-called particle-relabeling invariance is conventionally introduced to treat vorticity (Lin 1965). It is noted that, in the present fully Eulerian picture, there is no need to recourse to the particle picture in deriving the equation of motion and identifying the conserved quantities.

3. Incorporation of scalar field and an eddy viscosity model

3.1 *Role of imaginary part in the pseudo-action*

$\Phi(\boldsymbol{u})$ introduced in the previous section are traceless elements of $GL(2,C)$ but their set does not close with respect to multiplications because $\mathrm{Tr}\big(\Phi(\boldsymbol{u})\Phi(\boldsymbol{u}')\big)\neq 0$ in general. This may be an unsatisfactory feature if higher order interactions of more general forms are going to be taken into account.

This problem is resolved by removing the traceless constraint for Φ and incorporating a scalar field φ as

$$\Phi(\boldsymbol{u},\varphi)=\varphi\mathbf{1}+\boldsymbol{u}\cdot\boldsymbol{\sigma}, \tag{3.1.1}$$

where $\varphi(\neq 0)$ and $\boldsymbol{u}$ are generally complex. As before, we later let them be real. The set of $\Phi(\boldsymbol{u},\varphi)$ is closed and forms a group $GL(2,C)$ for $\det\Phi=\varphi^2-\boldsymbol{u}^2\neq 0$. The Lagrange multipliers μ_a and μ_a^* in (2.1.2) are now relegated at the beginning and $\Phi^\dagger$ is freely varied. We again obtain (2.1.18) but with (3.1.1). Noting that $\{\boldsymbol{\sigma},\Phi\}=2\varphi\boldsymbol{\sigma}+2\boldsymbol{u}$, (2.1.18) is now takes for real φ and $\boldsymbol{u}$ the form

$$\begin{aligned}&\dot{\Phi}+\frac{1}{2}(\varphi\boldsymbol{\sigma}+\boldsymbol{u})\cdot(\nabla\varphi+\nabla u_j\sigma_j)+\frac{1}{2}(\nabla\varphi+\nabla u_j\sigma_j)\cdot(\varphi\boldsymbol{\sigma}+\boldsymbol{u})\\&=\dot{\Phi}+\varphi\nabla\varphi\boldsymbol{\sigma}+\boldsymbol{u}\cdot\nabla\varphi+\boldsymbol{u}\cdot\nabla u_j\sigma_j+\varphi\nabla\cdot\boldsymbol{u}\\&=c_0\nabla^2\Phi+\mathbf{F}.\end{aligned}$$

Simply taking a trace after multiplying σ_i or a unit matrix to this equation, we obtain

$$\dot{u}_i+\boldsymbol{u}\cdot\nabla u_i=c_0\nabla^2u_j-\varphi\partial_i\varphi-\frac{\partial_i p}{\rho}+f_i \tag{3.1.2a}$$

$$\dot{\varphi}+\nabla\cdot(\varphi\boldsymbol{u})=c_0\nabla^2\varphi. \tag{3.1.2b}$$

Note that the N-S equations have been modified by the presence of a term $-\varphi\partial_i\varphi$ on r.h.s. of (2.1.2a). This is the reaction term of φ to $\boldsymbol{u}$ that emerges owing to the action of $\boldsymbol{u}$ to φ in (3.1.2b). The appearance of the action-reaction relation of this kind is the consequence of invoking the variational principle on the dynamics of Φ represented by (3.1.1). φ is something that is advected like a passive scalar when $\partial_i\varphi^2$ is sufficiently small. The meaning of φ will be elicited from interaction between φ and $\boldsymbol{u}$ that will be introduced in the followings.

New terms must be invariant under rotation and Galilei transformation. As the candidates of such terms, we look for the forms consisting of $\mathrm{Tr}\Phi$ and $\nabla\Phi$ and their hermitian conjugates. Consider $A^{\varphi}_{\mathrm{kin}}=\int \mathrm{L}^{\varphi}_{\mathrm{kin}}dtd\boldsymbol{r}$ where

$$\mathrm{L}^{\varphi}_{\mathbb{K}\mathrm{in}} = \frac{i\lambda_0}{8}\Big((\mathrm{Tr}\nabla\Phi^\dagger)^2 - (\mathrm{Tr}\nabla\Phi)^2\Big), \tag{3.1.3}$$

with $\lambda_0 > 0$. As before, the variation of A_{dif} is given by

$$\frac{\delta A^{\varphi}_{\mathrm{kin}}}{\delta\Phi^\dagger} = \frac{i\lambda_0}{4}\nabla^2\mathrm{Tr}\Phi^\dagger. \tag{3.1.4}$$

In the limit of $\mathrm{Im}\varphi = \mathrm{Im}\boldsymbol{u} = 0$, which is the solution of the equations of motion, $A^{\varphi}_{\mathrm{kin}}$ does not affect the equation for $\boldsymbol{u}$, while the equation (3.1.2) is modified to

$$\dot{\varphi} + \boldsymbol{u}\cdot\nabla\varphi = \lambda\nabla^2\varphi,\ \lambda \equiv \lambda_0 + c_0. \tag{3.1.5}$$

φ behaves as a diffusive passive scalar for $\lambda > 0$.

3.2 *Interaction and the physical meaning of φ*

There are an infinite number of possible interactions among Φ and $\Phi^\dagger$ with the invariance under transformations discussed in the previous section. In the followings, keeping the arguments in the previous subsection in mind, we construct as simple a Lagrangian as possible that gives physically acceptable interactions.

As the simplest third order interaction, we adopt $A^{(3)} = \int \mathrm{L}^{(3)}d\boldsymbol{r}dt$ where

$$\mathrm{L}^{(3)} = \frac{ic_1}{8}\Big(\mathrm{Tr}\Phi^\dagger\mathrm{Tr}(\nabla\Phi^\dagger)^2 - \mathrm{Tr}(\nabla\Phi)^2\mathrm{Tr}\Phi\Big) - \frac{ic_1}{16}\Big((\mathrm{Tr}\Phi^\dagger)(\nabla\mathrm{Tr}\Phi^\dagger)^2 - (\nabla\mathrm{Tr}\Phi)^2\mathrm{Tr}\Phi\Big). \tag{3.2.1}$$

The second term on r.h.s. of (3.2.1) is present to assure that higher order interactions between φ and $\boldsymbol{u}$ exist only when the velocity gradients exist. In fact, $\mathrm{L}^{(3)} = 0$ when $\boldsymbol{u} = 0$.

The variational contributions that follows from this term is given by

$$\mathrm{Tr}\left(\frac{\delta A^{(3)}}{\delta\Phi^\dagger}\sigma_i\right) \to -ic_1\nabla(\varphi\nabla u_i), \tag{3.2.2a}$$

$$\mathrm{Tr}\left(\frac{\delta A^{(3)}}{\delta\Phi^\dagger}\right) \to \frac{ic_1}{2}(\nabla\boldsymbol{u})^2, \tag{3.2.2b}$$

where $(\nabla\boldsymbol{u})^2 \equiv \partial_i u_j \partial_i u_j$. Arrows mean that all fields are real.

Next, we consider the real potential term for φ :

$$A_V = i\int \big(V(\mathrm{Tr}\Phi/2) - V(\mathrm{Tr}\Phi^\dagger/2)\big)d\boldsymbol{r}dt. \tag{3.2.3}$$

The meaning and the form of the function $V(x)$ will be argued later. Its variational contributions are

$$\mathrm{Tr}\left(\frac{\delta A_V}{\delta\Phi^\dagger}\sigma_i\right) = 0, \tag{3.2.4a}$$

$$\mathrm{Tr}\frac{\delta A_V}{\delta\Phi^\dagger} = -\frac{i}{2}V'(\mathrm{Tr}\Phi^\dagger/2). \tag{3.2.4b}$$

Adding the above terms to A_{NS} given by (2.1.17a) with Φ being replaced by the one defined in

(3.1.1), the pseudo-action we adopt is

$$A = A_{\mathrm{NS}} + A^{\varphi}_{\mathrm{kin}} + A^{(3)} + A_V. \tag{3.2.5}$$

The variational equations of motion that follows from A in real space of fields are

$$\dot{u}_i + \boldsymbol{u}\cdot\nabla u_i = \nabla\cdot\left((c_0 + c_1\varphi)\nabla u_i\right) - \frac{1}{2}\partial_i\varphi^2 - \frac{\partial_i p}{\rho} + f_i, \tag{3.2.6a}$$

$$\dot{\varphi} + \boldsymbol{u}\cdot\nabla\varphi = \lambda\nabla^2\varphi - \frac{c_1}{2}(\nabla\boldsymbol{u})^2 + \frac{1}{2}V'(\varphi). \tag{3.2.6b}$$

The first equation is a generalization of the N-S equation in which $c_0 + c_1\varphi$ acts on $\boldsymbol{u}$ as the viscosity coefficient. The second equation describes how φ is advected and diffuses via. shear stress and self-interaction.

The system (3.2.6) is essentially identical to the one derived by the variational principle with a non-holonomic condition and the action-reaction principle (Takahashi 2016). What is important is not whether the condition is holonomic or non-holonomic but whether the principle of action-reaction is fulfilled.

We notice that (3.2.6a) and (3.2.6b) constitute a structure similar to the eddy viscosity models. They hopefully close the Reynolds average equations to best approximation. By contrast, the equations (3.2.6) were generated from the stationary action principle in a dynamically consistent way. Because of its consistency and simplicity, the system (3.2.6) is worth further exploration. Unfortunately, the relation between φ in (3.2.6) and the eddy viscosity in eddy viscosity models is not yet clear. We may call any system derived from the stationary action principle like (3.2.6) a dynamical effective viscosity model (DEVM). (However, it should not be confused with the dynamic eddy viscosity model which varies the value of the model parameter during the numerical calculations and find the best one using some criterion. There, 'dynamic' is a notion in numerical analysis. See Germano et al. 1991 ; Lilly 1992 ; Park et al. 2006.)

3.3 *Form of $V(x)$ and minimal DEVM*

In order to infer the form of $V(x)$, we notice the invariance of the original N-S equation under the space-time inversion, $\boldsymbol{r}\to-\boldsymbol{r}$, $t\to-t$, $\boldsymbol{u}\to\boldsymbol{u}$, $\boldsymbol{f}\to-\boldsymbol{f}$ accompanied with the change of sign of the viscosity, $\nu\to-\nu$. Another noticeable invariance is for stationary flows. In this case, the equation is invariant under $\boldsymbol{r}\to\boldsymbol{r}$, $\boldsymbol{u}\to-\boldsymbol{u}$, $\boldsymbol{f}\to\boldsymbol{f}$ together with the sign change of viscosity (For an application of this invariance to finding vortex solutions, see Takahashi 2015). The *viscosity inversion invariance* in these cases implies $\varphi\to-\varphi$ must be a symmetry in DEVM. Therefore, $V'(\varphi)$ must be an even func-

tion of φ.

When the flow is stationary and at the same time the velocity gradients are entirely absent, the fluid is uniform and isotropic and the viscosity must be temporally and spatially constant. In this case, the equation must give $\varphi = \varphi_0 =$constant as the solution. Therefore, $V'(\varphi)$ is a regular function of $\varphi^2 - \varphi_0^2$. The features noted above are implemented in a simplest way into the form

$$V'(\varphi) = -c_2(\varphi^2 - \varphi_0^2) \text{ or } V(\varphi) = c_2\left(\varphi_0^2\varphi - \frac{1}{3}\varphi^3\right). \tag{3.3.1}$$

We rewrite the model derived above in terms of a dimensionless scalar $\phi \equiv \varphi/\varphi_0$ as

$$\dot{u}_i + \boldsymbol{u}\cdot\nabla u_i = \nu_0\,\nabla\cdot((c'_0+\phi)\,\nabla u_i) - \frac{1}{2}\partial_i\varphi^2 + \tilde{f}_i, \tag{3.3.2a}$$

$$\dot{\phi} + \boldsymbol{u}\cdot\nabla\phi = \lambda\nabla^2\phi - \frac{\nu_0}{2\varphi_0^2}(\nabla\boldsymbol{u})^2 - \frac{\lambda_1}{2}(\phi^2-1), \tag{3.3.2b}$$

where $\nu_0 \equiv c_1\varphi_0$, $c_0{}' = c_0/\nu_0$, $\lambda_1 \equiv c_2\varphi_0$. ν_0 is a representative kinematic viscosity. The configuration $\phi(\boldsymbol{r}) \equiv 1$ in the second equation implies the absence of the velocity gradient. We assume $\lambda_1 > 0$, so that this configuration is expected to be stable because a small deviation from $\phi = 1$ decays with the time constant $1/\lambda_1$.

In the above dynamical system, the field ϕ behaves and interacts with $\boldsymbol{u}$ in an analogous manner to the eddy viscosity in the averaged Reynolds-stress models. The flow equation (3.3.2a) is supplemented by an additional equation for ϕ. Therefore, the system is viewed as an example of OEEVMs, which have been developed by Spalart and Allmaras (1992, 1994). Their model employs the equation for the eddy viscosity ν_t like

$$\dot{\nu}_t + \boldsymbol{u}\cdot\nabla\nu_t = \frac{1}{\sigma}\nabla\cdot(\nu_t\nabla\nu_t) + \frac{c_{b2}}{\sigma}(\nabla\nu_t)^2 + c_{b1}S\nu_t - c_{w1}f_w\left(\frac{\nu_t}{d}\right)^2. \tag{3.3.3}$$

The third and the fourth terms on r.h.s. are the production and destruction terms, respectively. S is the characteristic rate of strain. f_w is a function introduced to adjust the outcome of the model to experimental result. Such a method of model construction was originally proposed by Van Driest (1956). Its modified versions have been commonly devised and employed in eddy viscosity models of various forms. Its role here is to suppress the destruction effect near the wall. The destruction term is of the square term of ν_t, so that it corresponds to the potential term in our model. Spalart and Allmaras's model (1992, 1994) reproduces the experimentally known mean-velocity profile of turbulent flow in the viscous sublayer and the logarithmic layer. In our model, however, the potential term turns out later to give positive contribution to $\dot{\nu}_t$ because of smallness of ϕ near the wall. The destruction term is provided by the gradient of $\boldsymbol{u}$ and ϕ. In the next section, we will see by numerical

calculations how effectively this destruction term works in our model.

4. Turbulent channel flow and pipe flow

In this section, we apply (3.3.2) to a turbulent channel flow with interpreting $\boldsymbol{u}$ as the mean velocity. Although the final form of the model treated below is identical to the one discussed previously (Takahashi 2016), we will repeat some arguments in order to highlight the consequence of our model that is novel and is conceptually distinctive from the prevalent eddy viscosity models.

We set $c'_0=0$ throughout our analyses given in the followings. Let us consider first a steady parallel flow $\boldsymbol{u}=(u_x(z), 0, 0)$ bounded by two planes at $z=0$ and $2d$ in the Cartesian coordinate. The system is assumed to be uniform in the x- and y-directions. The equations of motion read

$$(\phi\hat{u}_x{}')'+\alpha=0,\ \alpha\equiv\frac{\tilde{f}_x\lambda}{\rho\nu_0\varphi_0\lambda_1}=\frac{\mathrm{Re}}{\mathrm{Fr}^2}, \tag{4.1}$$

$$\phi''-\frac{\beta}{2}(\hat{u}'_x)^2-\frac{1}{2}(\phi^2-1)=0,\quad \beta\equiv\frac{\nu_0}{\lambda}=\mathrm{Pr}, \tag{4.2}$$

where the dimensionless velocity $\hat{u}_x\equiv u_x/\varphi_0$ has been introduced. The prime stands for a derivative with respect to the dimensionless coordinate $\hat{z}\equiv z/\ell_c$ where $\ell_c=(\lambda/\lambda_1)^{1/2}$ is the characteristic length. (4.1) is nothing but (3.3.1a) with $i=x$. The equations for $i=y$ and $i=z$ are $0=0$ and $\varphi\varphi'=-p'/\rho+f_z/\ell_c$, respectively. The latter one is used to determine the z-dependence of the pressure.

α is assumed to be constant. $\hat{u}_x$ away from the wall will take values of the order of unity. The continuity condition is automatically satisfied. (4.1) and (4.2) imply that the profile of channel flow under a given boundary conditions is essentially determined by α and β. $\mathrm{Re}=\ell_c\varphi_0/\nu_0$ is the Reynolds number and Pr the Prandtl number. $\mathrm{Fr}=(\rho\varphi_0^2/(f_x\ell_c))^{1/2}$ is the (generalized) Froude number, which is a measure of rapidity of the flow as compared to the wave velocity.

(4.1) is integrated once to yield

$$\hat{u}'_x=\frac{C_1-\alpha\hat{z}}{\phi}, \tag{4.3}$$

with an integration constant C_1. Due to symmetry, $\hat{u}'_x=0$ at the middle point of the channel, so that, from (4.3), C_1 is related to the half channel width d and the parameter α by

$$C_1=\alpha\hat{d}\equiv\alpha d/\ell_c. \tag{4.4}$$

Let ϕ_0 be the value of ϕ at the wall and write $\hat{u}_x\sim u_{1}\hat{z}$ (no-slip condition) and $\phi\sim\phi_0+\phi_1\hat{z}$ near the wall, so that, from (4.3), we have

$$\phi_0 u_1 = C_1. \tag{4.5}$$

u_x at the viscous sublayer is usually expressed as $u_x = (u_\tau/l_\tau)z$ or

$$\hat{u}_x = (\hat{u}_\tau/\hat{l}_\tau)\hat{z} = u_1\hat{z}. \tag{4.6}$$

u_τ is the wall-friction velocity, and l_τ the wall-friction length. Let us introduce a constant γ by

$$\hat{l}_\tau \equiv l_\tau/\ell_c = \gamma\phi_0. \tag{4.7}$$

Then, comparing (4.7) with (4.5) and (4.6), we also have

$$\hat{u}_\tau \equiv u_\tau/\varphi_0 = \hat{l}_\tau C_1/\phi_0 = \gamma C_1. \tag{4.8}$$

From (4.7) and (4.8), we see that, if $\gamma \sim O(1)$, $(\nu_0/\lambda_1)^{1/2}\phi_0$ and $\varphi_0 C_1$ respectively provide the measures for l_τ and u_τ.

Numerically integrating (4.1) and (4.2) is easy. Some examples of the solutions are shown in Fig. 1 and Fig. 2 as functions of z/l_τ . For values of parameters, see the figure captions. We have chosen the values of parameters so as for the equalities $\phi' = \hat{u}'_x = 0$ to hold at the midpoint.

Except for very vicinities of the wall, ϕ monotonically increases with z from very small value to a maximum at the middle point of the channel. In the region $z<d/4$, ϕ is well approximated by a linear function of z.

The velocity distributions are shown in Fig. 2. The best fitting to the experimental data of the

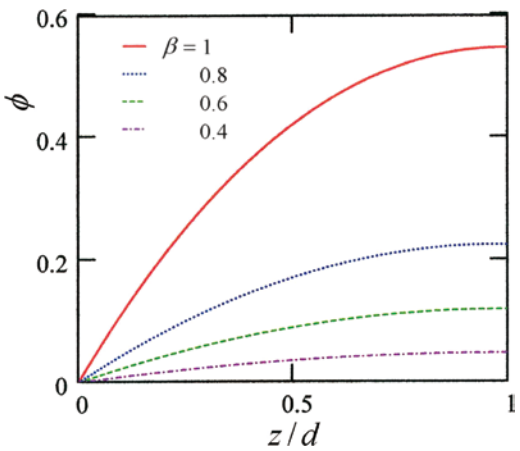

Fig. 1 ϕ vs. z/d for the Prandtl number $\beta = 1$ (solid curve), 0.8 (dotted curve), 0.6 (dashed curve), and 0.4 (dash-dotted curve). $\alpha = 0.001$ and $\phi_0 = 0.001$ are fixed. Other parameters $(\phi'(0), \hat{d})$ are taken as Solid curve : (−0.45, 1.7), Dotted curve : (−0.38, 0.99), Dashed curve : (−0.3, 0.7), Dash-dotted curve : (−0.2, 0.44).

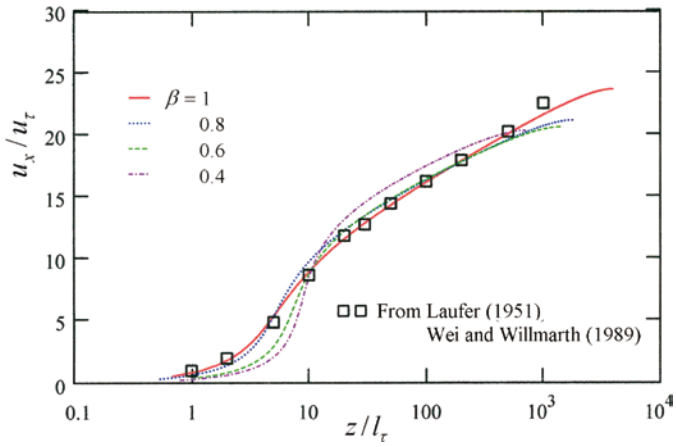

Fig. 2　Examples of $u_x/u_\tau (= \hat{u}_x/\hat{u}_\tau)$ vs. $z/l_\tau (= \hat{z}/\hat{l}_\tau)$ as the solutions of (4.1) and (4.2). The meanings of the curves are same as in Fig. 1. $(\hat{u}_\tau, \hat{l}_\tau)$: $(0.01, 4.3\times10^{-4})$ for solid curve, $(0.01, 5.6\times10^{-4})$ for dotted curve, $(0.01, 5\times10^{-4})$ for dashed curve, $(0.01, 6.3\times10^{-4})$ for dash-dotted curve. Squares are experimental data taken from Laufer (1951) and Wei and Willmarth (1989).

velocity distribution (Laufer 1951, Wei and Willmarth 1989) was sought for four values of β.

Fairly good agreement with the experimental observations is achieved for $\beta = 1$. In particular, the distinction between the regions in which u_x grows linearly $(z/l_\tau < 2)$ and almost logarithmically $(z/l_\tau > 30)$, which is characterized by the bending of curve in between, is clearly observed in Fig. 2. Although not explicitly shown here, the negativity of ϕ_1 seems quite effective to achieve this feature of the velocity distribution. Note that no adjustable function like Van Driest's damping functions was utilized to obtain this result.

The calculated velocity in the logarithmic layer, $80 < z/l_\tau < 800$, behaves as

$$\frac{\hat{u}_x}{\hat{u}_\tau} \approx \frac{1}{\kappa}\ln(\hat{z}/\hat{l}_\tau) + C \approx \frac{1}{0.36}\ln(\hat{z}/\hat{l}_\tau) + 6.8 \qquad (4.9)$$

That is, the Kármán constant κ in our model is about 0.36. Together with $C \approx 6.8$, these are to be compared with the experimentally known values $\kappa_{\mathrm{exp}} = 0.37 \sim 0.4$ and $C_{\mathrm{exp}} = 3.7 \sim 5$ (Laufer 1951, Wei and Willmarth 1989). Tuning of the boundary values to improve further the result will be possible. The origin of the logarithm-like behaviour (4.9) is the presence of a region of approximately linear growth of ϕ, $\phi \sim \phi_1' \hat{z}$, off the wall. Other powers for the functional form of ϕ in this region would be possible. In that case, power behaviour for the velocity distribution would emerge, as had been advocated by some authors (Barenblatt and Chorin 1993 ; Wosnik et al. 2000). Zanoun et al. (2004) have experimentally suggested the validity of the logarithmic behaviour and a value $1/e \approx 0.37$ for Kármán constant. In view of our present model, the distinction of this kind bears no physical sig-

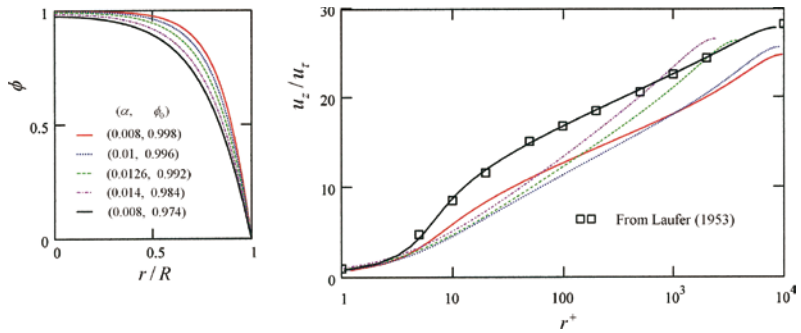

Fig. 3 Flow in a pipe. Left panel : ϕ as a function of r/R for various α and ϕ_0. R is the pipe radius and r is the radial distance from the central axis. The meanings of the curves are designated in the right panel. Right panel : Mean velocity in pipe flow for some parameter values of α and ϕ_0. $r^+ \equiv (R-r)/l_\tau$. $\phi_1 = 0.04$. $(\hat{R}, \hat{u}_{\max}) =$ (8.6, 0.53) for red solid curve ; (7.8, 0.5) for blue dotted curve ; (7, 0.46) for green dashed curve ; (6.2, 0.014) for purple dot-dashed curve ; (5.7, 0.37) for black solid curve. Squares are experimental data taken from Laufer (1953).

nificance.

The system (3.3.2) can also be applied to a flow in a circular pipe. Write $\boldsymbol{u} = (0, 0, u_z(r)$ in the cylindrical coordinate with r being the distance from the central axis of the pipe. $\beta = 1$ is fixed. The result is shown in Fig. 3. ϕ is almost constant in the central region of the pipe and decreases to a very small value at the pipe wall. A smaller ϕ gives rise to a larger second derivative of the velocity and in turn more rapid decreases of the velocity particularly in the viscous sublayer as is shown in Fig. 3. Consequently, larger deviation of ϕ_0 from unity brings about larger bending of the curve of u_z in the transition region. The Kármán constant decreases as α increases. The conformity with the experiments (Laufer 1953, Ferro 2012) is quite well for $\alpha = 0.008$ and $\phi(0) = 0.974$.

The conclusion of this section is that the minimal DEVM derived from the stationary action principle can be a model of mean turbulent flows. The equations of motion derived in this model are equivalent to the ones previously constructed by a variational method with a non-holonomic condition (Takahashi 2016). The solutions of the equations of motion, therefore, coincide with the ones in the present minimal DEVM. In particular, the Reynolds stress in parallel flow is given in Takahashi (2016), which was calculated with help of the N-S equations.

5. Summary and remarks

We showed that constructing of action, which we called the pseudo-action, within the Eulerian

description of motion of viscous fluid is possible in terms of complex matrix representation of fields. The required symmetries of the pseudo-action were of space-time translation, rotation and Galilei transformation. By appropriately choosing the kinematic and interaction terms, the stationary action principle produces the field equations equivalent to the N-S equation or the eddy viscosity model according to whether the field is traceless or not. The conserved 'Hamiltonian', 'momentum' and 'angular momentum' derived from the pseudo-action of the real fields all trivially vanish. Nontrivial conservation of circulation at infinity was proved in the complex space of the fields.

The derived DEVM, when applied to a channel flow, provides a set of equations for the mean velocity and the eddy viscosity with two free model parameters, one is the Prandtl number and the other a ratio of Reynolds number to the Froude number squared. By appropriately choosing the boundary conditions together with the model parameters, in particular, $\mathrm{Pr} \approx 1$, DEVM gave fairly nice agreements with experiments. Considering the model's simplicity, this may be due to the fact that the stationary action principle enables us to construct dynamics incorporating correctly related interactions between velocity and viscosity.

DEVM is expressed in terms of a scalar matrix Φ and describes the dynamics of mean velocity and eddy viscosity. Is it possible to incorporate fluctuations like the Reynolds stress? In order to answer this question, let us recall that the fundamental requirement for DEVM is to fulfil a few invariance principles, namely, invariance under translation, rotation and Galilei transformation together with the viscosity-inversion invariance. Therefore, inclusion of fluctuation, if it is tensors, may be done by introducing a vector matrix, say $R_i = R_{ij}\sigma_j$ and by writing down an invariant action. This possibility will be worth a detailed research. If the result is affirmative, such a model may be called a dynamical eddy viscosity model.

In the framework of Eulerian field theory, any dynamically consistent extension of our minimal model with the symmetries being unbroken will be straightforward. There exist an infinite number of interactions that are consistent with the requirements mentioned above. We do not yet find a general rule of selecting a priori physically preferable ones. Nevertheless, that $\mathrm{Pr} \approx 1$ yields a best fitting tells us an important hint. $\mathrm{Pr} \approx 1$ states that the rate of the energy dissipation due to shear stress is nearly equal to the rate of diffusion of effective viscosity field. The diffusion of the effective viscosity field may be regarded to correspond to the Kolmogorov cascade in turbulence that is responsible for the established scaling law of energy dissipation. As long as the mean profile of turbulence is concerned, therefore, the role of eddies is expected to dominate over other (i.e., molecular) elements irrespective of the scale of the system. It is thus prompted to apply our model in its present form to

other, possibly astronomical, systems of entirely different scales. The results of study in this area will be reported in near future.

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