【Article】

Vorticity Equation, Current Conservation and the Solutions of the Navier–Stokes Equation

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Abstract The motion of a Newtonian fluid is described by the Navier–Stokes equation, which is a partial differential equation for the velocity field with respect to space and time variables. Its nonlinearity is the source of difficulty in finding general solutions. In this paper, a method of determining the velocity fields in the Navier–Stokes equation is proposed by noting a similarity of the vorticity equation in two dimensions to the current conservation equation. By noting this correspondence, the original Navier–Stokes equation for flows of one degree of freedom is transformed to a linear partial differential equation with respect to space variables only. Some exact two–dimensional solutions including well–known ones are derived by this method. The solutions for swirling flows together with its perturbation can be a model of typhoon.

Key words: Navier–Stokes equation, continuity equation, exact solutions, typhoon

1. Introduction

The dynamics of the Newtonian fluid is governed by the Navier–Stokes equation, the continuity equation and the equation of state. The first two are the non–linear partial differential equations with respect to space and time. Together with the absence of any internal symmetry, finding 'exact' solutions in a general way is very difficult. Many exact solutions have been found so far by imposing various physical requirements on the boundary conditions, the degrees of freedom and the global properties of the fluid and its motions. Incompressibility is one of the conditions customarily adopted in literatures. For a review, see Wang (1991) and Drazin and Riley (2006).

On the other hand, owing to the development of the technique of numerical analyses, numerical solutions of ordinary differential equation are now easily obtained with grate accuracies. Therefore, from a practical point of view, we may also regard transforming the Navier–Stokes equation to the ordinary differential equations as equivalent to obtaining exact solutions.

Ever since the Navier–Stokes equation was discovered and studied by Navier (1827), Poisson (1831), Saint–Venant (1843) and Stokes (1845), many efforts have been devoted to find exact solutions or the ordinary differential equations under peculiar boundary conditions.　For rotational stream in two-dimension that we are interested in this paper, Kampe de Feriet (1930, 1932) and Tsien (1943) have found various exact solutions.　For a review, see Wang (1991).

One of the origins of difficulty of solving the Navier-Stokes equation lies in that one has to take account of the continuity equation separately.　This situation is in contrast with the case of U(1) symmetric quantum mechanics or field theories, where the continuity of conserved quantity is automatically fulfilled by the solution of the Schrödinger equation or field equations.　If one can take advantage of such a property of the U(1) symmetric theories in solving the Navier-Stokes equation, finding its solutions may be greatly helped.

In this paper, we focus our attention to the exact solutions of the Navier-Stokes equation.　By the 'exact' solution, we here mean those that are expressed in terms of the well-known analytic functions, ordinary differential equations or integrations to be easily performed by the numerical techniques. We shall present a method of finding exact two-dimensional solutions of the Navier-Stokes equation by utilizing the property of the conserved current.　Such currents may be explicitly constructed, e.g., in the framework of any U(1) symmetric theories.　The point we are going to notice is a mathematical parallelism between the vorticity equation and the current conservation equation.　This parallelism enables us to transmute a conserved current to the vorticity via an ordinary differential equation. The method will be shown to be entirely consistent with directly solving the Navier-Stokes equation. The customary constraint of incompressibility will generally be removed throughout our discussions.

In the next section, we elaborate the idea that leads to the transmutation equation.　In sec.3, some applications of the transmutation equation are presented.　In sec.4, a way of extension of the method to higher degrees of freedom is discussed.　Sec.5 is devoted to a summary.

2.　Correspondence of the vorticity equation and the current conservation equation for the solenoidal fluid

2.1　The Navier-Stokes equation

The Navier-Stokes equation is the partial differential equation for the velocity field and is written as follows :

$$\dot{v}+(v\cdot\nabla)v=\frac{\nu}{3}\nabla(\nabla\cdot v)+\nu\nabla^2 v-\frac{1}{\rho}\nabla P+f. \tag{2·1·1}$$

v, ρ , P and f are the velocity field, mass density, pressure and external volume force, respectively.

They are all functions of the space and time. We begin with the solenoidal velocity derived from a vector field $\boldsymbol{A}$ by

$$v = \nabla \times \boldsymbol{A}. \tag{2·1·2}$$

Furthermore, the force is assumed to be a conserved one, $\boldsymbol{f} = -\nabla U(\boldsymbol{r})$, where $U(\boldsymbol{r})$ is the potential at spatial position $\boldsymbol{r}$. ν is the kinematic viscosity. The dot on a variable denotes the partial differentiation with respect to time t.

The continuity equation for the mass is given by

$$\dot{\rho} + \nabla(\rho v) = 0. \tag{2·1·3}$$

For incompressible fluids, (2·1·3) leads to the familiar constraint

$$\nabla \cdot v = 0. \tag{2·1·4}$$

Notice that (2·1·4) alone does not assure the incompressibility, i.e., $\dot{\rho} = \nabla\rho = 0$. Instead, (2·1·4) leads to

$$\dot{\rho} + \nabla\rho \cdot v = 0. \tag{2·1·5}$$

For the pressure P, which is determined by the equation of state and the external force, we adopt the form

$$P = P_f + P_\rho,\ P_\rho = C\rho^\gamma. \tag{2·1·6}$$

P_f on the r.h.s. is due to the external force and will be expressed as a direct function of spatial coordinate. For the compressible fluid, this term is balanced by the external force. P_ρ is determined by the equation of state. For the incompressible fluid, P_ρ is a mere constant. (The pressure P generally depends on the temperature too. The 'constant' C may be variable due to such dependences.)

2.2 The equation of vorticity

Consider the rotational fluid defined by (2·1·2). For $\boldsymbol{A}$, we assume the form

$$\boldsymbol{A} = (0, 0, A_z(x, y)) \qquad \text{or} \qquad \boldsymbol{A} = (0, 0, A_z(r, \theta)) \tag{2·2·1}$$

The former is in the Cartesian coordinate (x, y, z) and the latter is in the cylindrical coordinate (r, θ, z). The velocity components are given by

$$v_x = \frac{\partial A_z}{\partial y},\ v_y = -\frac{\partial A_z}{\partial x}, \tag{2·2·2a}$$

$$v_r = \frac{1}{r}\frac{\partial A_z}{\partial \theta},\ v_\theta = -\frac{\partial A_z}{\partial r}, \tag{2·2·2b}$$

and v_z=0.　Note that $\nabla\cdot\boldsymbol{A}=0$ and $\nabla\cdot v=0$.　Such an A_z is called the stream function.

The Navier–Stokes equation (2·1·1), when operated by $\nabla\times$ on the both sides, yields for conservative force

$$\nabla\times(\rho^{-1}\nabla P)\big|_x=\nabla\times(\rho^{-1}\nabla P)\big|_y=0, \tag{2·2·3a}$$

$$\dot{\zeta}+\nabla(\zeta v)=\nabla\nu\cdot\nabla\zeta+\nu\nabla^2\zeta-\nabla\times(\rho^{-1}\nabla P)\big|_z. \tag{2·2·3b}$$

Here, ζ is the z component of the vorticity, i.e., $\omega\equiv\nabla\times v=(0,0,\zeta)$ defined by

$$\zeta=\nabla\times v\big|_z=-\nabla^2 A_z. \tag{2·2·4}$$

(2·2·3a) means that the vectors $\nabla\rho$ and ∇P lie in the xy plane.　This will be assured if ρ and P do not depend on z.

The last term on the r.h.s. of (2·2·3b) identically vanishes for incompressible fluids.　For compressible fluids, this term may be replaced by $\nabla\times(\rho^{-1}\nabla P_\rho)\big|_z$, which also vanishes because P_ρ is a function of ρ only.　Therefore, hereafter we always drop this term in (2·2·3b)

2.3　Correspondence of the vorticity equation to the current conservation

Suppose that we have a set of a density ρ_c and a current $\boldsymbol{j}_c$ that obey the continuity equation

$$\dot{\rho}_c+\nabla\cdot\boldsymbol{j}_c=0. \tag{2·3·1}$$

Their space–time dependences are also supposed to be known.　This is always possible by choosing an arbitrary vector $\boldsymbol{j}_c(t,\boldsymbol{r})$ and defining the density by $\rho_c(t,\boldsymbol{r})=-\int^t\nabla\cdot\boldsymbol{j}_c(t,\boldsymbol{r})dt.$.　One may borrow their forms from other branch of physics.　For example, in quantum mechanics, these quantities are constructed from the wave function Ψ by

$$\rho_c=\Psi^*\Psi, \tag{2·3·2a}$$

$$j_c=ia((\nabla\Psi^*)\Psi-\Psi^*\nabla\Psi)=2a\rho_c\nabla\delta, \tag{2·3·2b}$$

where δ is the phase of Ψ.　a is a parameter appearing in the Schrödinger equation

$$i\hbar\dot{\Psi}=(-\hbar^2 a\nabla^2+U(\boldsymbol{r}))\Psi. \tag{2·3·3}$$

a=1/2m with the particle mass m.　$\hbar$ is the Planck's constant divided by 2π.　U is the potential. In quantum mechanics, ρ_c is interpreted as the probability of the particle to exist at a given space and time.

The prescription to find Ψ has been established, owing to the linearity of the Schrödinger equation. The space–time dependences of ρ_c and $\boldsymbol{j}_c$ are then explicitly known.　In our discussions, we regard

ρ_c and $\boldsymbol{j}_c$ (or δ) as known functions of space and time, although these quantities are not directly related to the corresponding counterparts in the classical fluid dynamics.

Now, we rewrite (2·3·1) as

$$\dot{\rho}_c + \nabla\cdot(\rho_c v) = \nabla\cdot(\rho_c v - \boldsymbol{j}_c) \tag{2·3·4}$$

v is the velocity field of the fluid we are considering. Here, we note the similarity of the l.h.s. of this equation to the one in (2·2·3b) for the vorticity. If the vorticity ζ is represented as a function of some ρ_c satisfying the continuity (2·3·1), then (2·2·3b) will be transmuted to the one that determines ζ in terms of ρ_c. We are thus lead to assume the form

$$\zeta = \zeta(\rho_c(t, \boldsymbol{r})). \tag{2·3·5}$$

In this case, $\dot{\zeta} = \zeta' \dot{\rho}_c$, $\nabla\zeta = \zeta' \nabla\rho_c$ (ζ', ζ'' etc. denote the differentiations of ζ with respect to ρ_c.), and the equation (2·2·3b) is rewritten as

$$(\dot{\rho}_c + \nabla\cdot(\rho_c v))\zeta' = (\nabla\nu\cdot\nabla\rho_c + \nu\nabla^2\rho_c)\zeta' + \nu(\nabla\rho_c)^2\zeta''. \tag{2·3·6}$$

In the above equation, the term involving the temporal differentiation can be eliminated by using (2·3·4). Thus, we have a linear differential equation

$$\nu(\nabla\rho_c)^2\zeta'' + (\nabla\nu\cdot\nabla\rho_c + \nu\nabla^2\rho_c - \nabla\cdot(\rho_c v - \boldsymbol{j}_c))\zeta' = 0. \tag{2·3·7}$$

v on the l.h.s. is related to ζ by (2·2·4), so that its spatial variation will also emerges through ρ_c. Then, together with the relations

$$\nabla\times v\,|_z = \partial_x\rho_c v'_y - \partial_y\rho_c v'_x = \zeta, \tag{2·3·8a}$$

$$\nabla\cdot v = \partial_x\rho_c v'_x + \partial_y\rho_c v'_y = 0, \tag{2·3·8b}$$

we have obtained a sufficient set of equations to determine three unknown functions, ζ, v_x and v_y. The integration constants and associated functions, if any, must be determined by invoking the original Navier-Stokes equation and the continuity equation.

The extension to the case in which ζ involves more than one functions as $\zeta(\rho_{c1}, \rho_{c2}, \cdots)$ is straightforward. The equation corresponding (2·3·7) takes on the form

$$\nu\sum_{j,k}\nabla\rho_{cj}\cdot\nabla\rho_{ck}\partial_j\partial_k\zeta + \sum_j\left(\nabla\nu\cdot\nabla\rho_{cj} + \nu\nabla^2\rho_{cj} - \nabla\cdot(\rho_{cj}v - \boldsymbol{j}_{cj})\right)\partial_j\zeta = 0 \tag{2·3·9}$$

where ∂_j stands for the derivative with respect to ρ_{cj}. Our considerations will be mostly focused on the case of the single variable, i.e., the one degree of freedom. An application of (2·3·9) will be given in sec.4.

2.4 Transmutation equation

(2·3·7) is homogeneous and we can proceed further with our arguments. Define a vector $\boldsymbol{Y}$ by

$$\nabla\cdot\boldsymbol{Y}\equiv\nu(\nabla\rho_c)^2\frac{\zeta''}{\zeta'}, \tag{2·4·1}$$

and rewrite (2·3·7) as

$$\nabla\cdot(\rho_c v-\boldsymbol{j}_c)=\nabla\cdot(\nu\nabla\rho_c+\boldsymbol{Y}). \tag{2·4·2}$$

By integrating this equation with use of (2·3·2b), v is expressed as

$$v=2a\nabla\delta+\nu\nabla\ln\rho_c+\rho_c^{-1}\boldsymbol{Y}-\nabla\rho_c^{-1}\times\boldsymbol{X}+\nabla\times(\rho_c^{-1}\boldsymbol{X}) \tag{2·4·3}$$

where $\boldsymbol{X}$ is an arbitrary vector. We assume it is a function of x and y (or r and θ in the cylindrical coordinate). On the other hand, from (2·1·2), the purely rotational vector v must be equal to the purely rotational term on the l.h.s. of (2·4·3). Thus, we have

$$\boldsymbol{A}=\rho_c^{-1}\boldsymbol{X}. \tag{2·4·4}$$

Here, a possible gradient term of a scalar function is omitted for simplicity, so that the components of $\boldsymbol{X}$ other than X_z are zero. The remaining term in (2·4·3) must vanish :

$$2a\nabla\delta+\nu\nabla\ln\rho_c+\rho_c^{-1}\boldsymbol{Y}-\nabla\rho_c^{-1}\times\boldsymbol{X}=0. \tag{2·4·5}$$

Multiplying ρ_c on the both sides of (2·4·5) and taking divergences, we have

$$\begin{aligned}\nu(\nabla\rho_c)^2\frac{\zeta''}{\zeta'}&=\nu\nabla\rho_c\cdot\nabla\ln\zeta'\\&=-\nabla\cdot(\nu\nabla\rho_c+j_c)+\nabla X_z\times\nabla\ln\rho_c\,|_z.\end{aligned} \tag{2·4·6}$$

These equations, which do not involve time-derivatives, tell us how the quantities ρ_c and δ can be transmuted to the fluid dynamical quantity, here the vorticity, in a manner consistent to the vorticity equation. In the case X_z can be chosen as a function of ρ_c only, then the last term of the r.h.s. of (2·4·6), which we call the X_z term, vanishes and the equation becomes quite tractable. On the other hand, interesting phenomena take place when the X_z term plays a nontrivial role, as we shall see later.

Two comments are in order. First, when ρ_c and $\nabla\cdot\boldsymbol{j}_c$ identically vanishes, and X_z is assumed to be a function of ρ_c, (2·4·6) trivially recovers the original equation (2·2·3b) for ζ. Non-triviality is manifested when ρ_c and $\boldsymbol{j}_c$ are time-dependent. Second, in (2·4·6), not only ρ_c but $\boldsymbol{j}_c$ too appears explicitly. Our assumption was that ζ acquires the coordinate dependence through ρ_c only. Therefore, in order to determine ζ according to (2·4·6), the integration must be performed under the condition $\boldsymbol{j}_c$=constant, except the cases in which $\boldsymbol{j}_c$is also a function of ρ_c.

2.5 Numerically solving the transmutation equation

We here consider a system with one degree of freedom and all physical quantities are functions of x and t only. Since the functional forms of $\rho_c(x,t)$ and $\boldsymbol{j}_c(x,t)$ are supposed to be explicitly known, the equation (2·4·6) is easily solved numerically. From the initial condition $\zeta(0,0)=\zeta_0$ and $\zeta'(0,0)=\zeta_1$, one can evaluate the values of ζ at the vicinity of $x=0$, $t=0$ from the rule

$$\zeta(\Delta x,0)=\zeta(0,0)+\zeta'(0,0)\partial_x\rho_c\,\Delta x,\quad \zeta'(\Delta x,0)=\zeta'(0,0)+\zeta''(0,0)\,\partial_x\rho_c(0,0)\Delta x$$

The values of $\zeta(x,0)$ is determined by repeating this calculation in the x-direction. Similarly, from the values of $\zeta(0,\Delta t)$ and $\zeta'(0,\Delta t)$ that are calculated by knowing $\dot{\rho}_c(0,0)$, $\zeta(x,\Delta t)$ is determined. Finally, the velocity field is determined by integrating the equation

$$\nabla^2 v=-\nabla\times\omega. \tag{2·5·1}$$

As an example, let us take the forms

$$\rho_c(x,t)=\left(\frac{1}{\omega}\sin\omega t+at+b\right)e^{kx},\quad \nabla\cdot\boldsymbol{j}_c(x,t)=-(\cos\omega t+a)e^{kx}. \tag{2·5·2}$$

Obviously, v_x vanishes and the continuity holds. The result for $\nu=1$, $\omega=1$, $a=0.5$, $b=1$ and $k=-0.2$ is given in Fig.1. The ρ_c dependences of ζ in Fig. 1(a) is read out from this result by noting the one-to-one correspondence of x and ρ_c at each t. By integrating the result for ζ, we have the solution for v_y as is shown in Fig. 1(b). At any instant, the profile of v_y is parabolic and is similar to that of the Couette-Poiseuille flow.

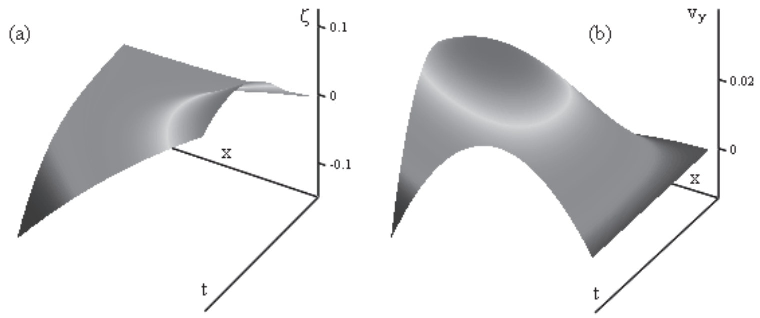

Fig. 1 Numerical solution to (2·4·6) for the input density and current (2·5·2) in $1<x<2$, $1<t<2.5$. (a) $\zeta(x,t)$. The initial condition is $\zeta(1,1)=0, \zeta'(1,1)=0.1$. (b) $v_y(x,t)$. The boundary condition is $v_y(1,t)=0$, $v_x=v_z=0$.

3.　Exact solutions to (2・4・6)

In this section, we give three examples in which exact steady solutions are found from (2·4·6) with no reliance on a concrete functional form of ρ_c except for that $\partial_x \rho_c$ does not identically vanish.　The reason is explained in the previous section.　In addition, one time-dependent example will be given.

Example 1 : $\nabla\rho = \nabla\delta = 0$ and v is dependent on x only.

In this first example, we elaborate the procedure of finding the solution.　Let us assume that the X_z term in (2·4·6) vanishes.　Integration of (2·4·6) in x yields $\ln \zeta' = -\ln \partial_x \rho_c$, which implies $\partial_x \rho_c \zeta' = \partial_x \zeta =$ constant.　(We use the symbol of partial derivative for easiness to see even for functions of a single variable.)　We readily have

$$\zeta = c_1 x + c_2 \tag{3·1·1}$$

with two integration constants c_1 and c_2.　The stream function A_z is obtained by solving Poisson equation (2·2·4) together with some boundary conditions.　If there is no boundary, then, from (2·2·4) and (2·1·2) we have

$$A_z = -\frac{c_1}{6}x^3 - \frac{c_2}{2}x^2 + c_3 x + c_4 y, \tag{3·1·2a}$$

$$v_x = \partial_y A_z = c_4,\ v_y = -\partial_x A_z = \frac{c_1}{2}x^2 + c_2 x - c_3. \tag{3·1·2b}$$

In the above derivation, we required that v is dependent on x only.　Inserting (3·1·2) to the Navier-Stokes equation (2·1·1) yields

$$-\frac{1}{\rho}\partial_x P + f_x = 0, \tag{3·1·3a}$$

$$-\nu c_1 - \frac{1}{\rho}\partial_y P + f_y = -c_4(c_1 x + c_2) \tag{3·1·3b}$$

These equations are satisfied when c_4 and all of ∇P, $\boldsymbol{f}$ and ρ are constant.　The continuity equation (2·1·5) is also fulfilled.　This is the Couette-Poiseuille's solution.　Note that the derivation of this solution does not dependent on the form of ρ_c.

Example 2 : $\nabla\rho = 0$, $\nabla\delta = (k, 0, 0)$ (the wave number k is constant.) and v is dependent on x only

Let us assume that the X_z term in (2·4·6) vanishes.　As in example 1, we have

$$\ln\zeta' = -\frac{2ak}{\nu}x - \ln \partial_x \rho_c \tag{3·2·1}$$

ζ is solved as

$$\zeta = c_1 + c_2 e^{c_3 x} \tag{3·2·2}$$

where we have made a redefinition by $c_3=2ak/v$. The stream function and the velocity field are given by

$$A_z = -\frac{c_1}{2}x^2 - \frac{c_2}{c_3^2}e^{c_3 x} + c_4 x + c_5 y \tag{3·2·3a}$$

$$v_x = c_5, \quad v_y = \frac{c_2}{c_3}e^{c_3 x} + c_1 x - c_4. \tag{3·2·3b}$$

The density and the pressure gradient are constant. This is the generalized Couette-Poiseuille's solution, which describes a flow between two plates, one of which is sliding to the y direction (Couette 1890). The constants in (3·2·3) are expressed in terms of ∇P, ν, ρ, together with the average flow velocity and the sliding velocity of a plate (Drazin and Riley 2006).

Example 3 : Axially symmetric flow

i) Time-independent solution

The case of the steady concentric flows with no boundary is considered here to show that (2·4·6) is in fact consistent with the Navier-Stokes equation. As a byproduct a new solution will be presented.

We adopt the cylindrical coordinate $\boldsymbol{r} = (r, \theta, z)$ and $v = (v_r, v_\theta, v_z)$ with $v_z=0$. The phase term on the r.h.s. of (2·4·6) vanishes. The assumption is that ζ is a function of r only. The general form of X_z may be given by

$$X_z = \nu\beta(r)\theta. \tag{3·3·1}$$

X_z itself is not a physical observable and can be multi-valued. Factoring out the constant ν is for convenience. Then, (2·4·6) takes on the form

$$\partial_r\rho_c\partial_r \ln\zeta'(r) = -\partial_r^2\rho_c - \frac{1}{r}\partial_r\rho_c - \frac{\beta\partial_r\rho_c}{r\rho_c}. \tag{3·3·2}$$

This can be solved as

$$\ln\zeta'(r) \equiv -\int^r dr\left(\frac{1}{r} + \frac{\beta}{r\rho_c}\right). \tag{3·3·3}$$

The stream function and the velocity field are given by

$$A_z = -\int^r \frac{dr}{r}\int^r drr\zeta(r) + h(\theta)\ln r, \quad h(\theta) = h_1\theta. \tag{3·3·4}$$

$$v_r = h_1\frac{\ln r}{r}, \tag{3·3·5a}$$

$$v_\theta = \frac{1}{r}\int^r drr\zeta(r) - \frac{h_1\theta}{r}. \tag{3·3·5b}$$

The Navier–Stokes equation in the cylindrical coordinate is

$$\dot{v}_r + v_r\partial_r v_r - \frac{v_\theta^2}{r} = \nu\left(\nabla^2 v_r + \frac{1}{3}\partial_r\nabla\cdot v - \frac{v_r}{r^2} - \frac{2}{r^2}\partial_\theta v_\theta\right) - \frac{\partial_r P}{\rho} + f_r, \tag{3·3·6a}$$

$$\dot{v}_\theta + v_r\left(\partial_r v_\theta + \frac{v_\theta}{r}\right) + \frac{v_\theta}{r}\partial_\theta v_\theta = \nu\left(\nabla^2 v_\theta + \frac{1}{3r}\partial_\theta\nabla\cdot v - \frac{v_\theta}{r^2}\right) - \frac{\partial_\theta P}{r\rho} + f_\theta. \tag{3·3·6b}$$

The time-derivative terms can be dropped here. The consistency requirement of these equations yields $\partial_\theta P = 0$ and $f_\theta = 0$. Let $h_1 \to 0$ in order to get rid of the θ-dependence in $v_{\hat{\theta}}$, while $c \equiv -\nu/h_1$ being kept fixed. By substituting v_θ given by (3·3·5b) to (3·3·6b), we have

$$\zeta(r) = \frac{c_1}{r^2}\int^{\ln r} dse^{-s^2/2c+2s}, \tag{3·3·7}$$

Or, equivalently, we can write the differential equation for v_θ

$$v_\theta'' \frac{3}{r}v_\theta' + \frac{1}{r^2}v_\theta = \frac{c_1}{r}e^{-(\ln r)^2/2c}. \tag{3·3·8}$$

c_1 is an arbitrary constant. Finite solutions are possible when c is positive.

$\beta(r)$ introduced in (3·3·2) is determined by differentiating (3·3·3) with r. The velocity field is determined in an independent way to ρ_c, although β depends on ρ_c. The continuity equation is satisfied if ρ is constant or a function of r only. Thus, the consistency of the transmutation equation (2·4·6) to the Navier–Stokes equation in this problem has been explicitly shown.

Numerical solutions are obtained by integrating (3·3·8) with a boundary condition $v_\theta = 0$ at $r = 0$ and are shown in Fig. 2. These solutions are intriguing in two points. First, they have no singularity and exhibit behaviours different from the well-known steady concentric flows that are singular at $r = 0$ or divergent at $r = \infty$ (Oseen 1911, Hocking 1963). This solution for the inviscid flow is never obtained from the Euler equation where the kinematic viscosity is set zero at the outset. Second, the

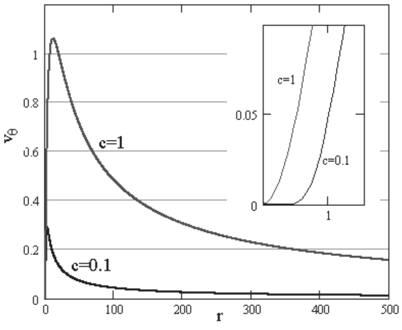

Fig. 2　r-dependences of v_θ as solutions of (3·3·8) with $c_1 = 0$ in arbitrary scales. The inset is for $0.2 < r < 1.5$.

azimuthal velocity near the symmetry axis takes very small values, rises rapidly as r increases, reaches a maximum at a certain radius and subsides gradually beyond it, thereby forming an 'eye' at the center. This profile reminds us of the one observed in the horizontal velocity distribution of typhoons (see, e.g., Emanuel 2004, Holland et al 2010 and references cited therein, Takahashi 2012). There is a mathematical proof which shows the long-term existence of three dimensional solutions that do not diverge but swirls slowly near $r = 0$ (Zadrzyńska and Zajączkowski 2009).

ii) Perturbation

The solution presented above has a flow profile quite similar to the ones used in the phenomenology of typhoon, so that it may be of a matter of interest to inquire what kind of perturbation is allowed around the solution. Let the radial and the azimuthal components are perturbed as $v_r \to v_r + \delta v_r = \delta v_r$, $v_\theta \to v_\theta + \delta v_\theta$. Substituting these in (3·3·6a) and (3·3·6b) and linearlizing the equations in δv_r and δv_θ, we have

$$\delta\dot{v}_r - \frac{2v_\theta \delta v_\theta}{r} = -\delta\left(\frac{\partial_r P}{\rho}\right), \tag{3·3·9a}$$

$$\delta\dot{v}_r + \left(\partial_r v_\theta + \frac{v_\theta}{r}\right)\delta v_r + \frac{v_\theta}{r}\partial_\theta \delta v_\theta = -\frac{1}{r}\delta\left(\frac{\partial_\theta P}{\rho}\right), \tag{3·3·9b}$$

where uses have been made of $v = 0$, $\partial_\theta v_\theta = 0$ and $\partial_\theta P = 0$. These equations relate the variations in the pressure, the density, δv_r and δv_θ. Let us assume that the density variation and the resultant pressure variation are small and the r.h.s. of each equation can be neglected. Then, the perturbations are expressed by sinusoidal functions

$$\delta v_\theta = A\sin\left[n\left(\frac{v_\theta}{r}(t-t_0)-\theta\right)\right], \tag{3·3·10a}$$

$$\delta v_r = \frac{2v_\theta}{r}\int^t \delta v_\theta dt = -\frac{2A}{n}\cos\left[n\left(\frac{v_\theta}{r}(t-t_0)-\theta\right)\right], \tag{3·3·10b}$$

where n is an integer and designates the mode of oscillation. This expression is valid for large n limit (see Appendix). δv_r of high modes will be neglected. Various kinds of perturbations are observed by varying the choice of parameters. One example of the temporal and spatial dependences of $v_\theta + \delta v_\theta$ are shown in Fig. 3 for $c = 0.2$, $n = 5$. In this example, additional maxima of velocity emerge and migrate inward until they merge together. An analogous phenomenon has been observed in the evolution of typhoon (Willoughby et al. 1982). Although not shown here, after t_0, the single peak splits to several ones, some of which gradually move outward.

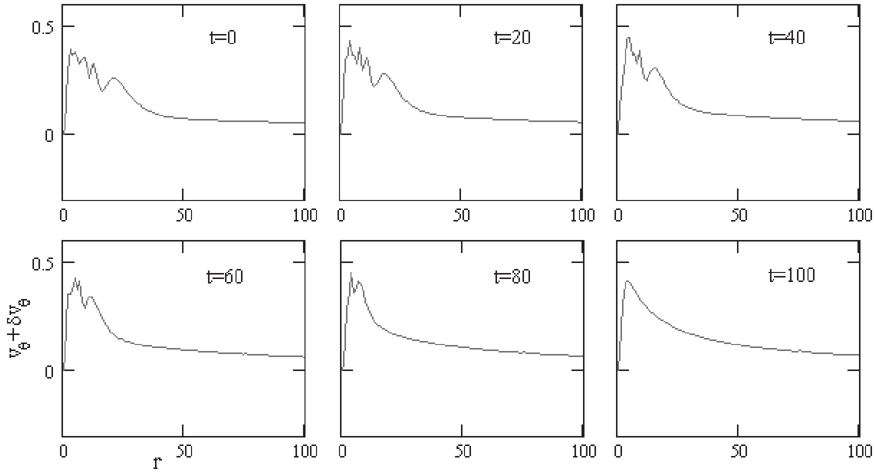

Fig. 3　Temporal variation of the perturbed azimuthal velocity $v_\theta + A\sin[n((v_\theta/r)t - t_0 - \theta)]$ from $t = 0$ to 100 for $c = 0.2$, n=1, $t_0 = 100$ and $A = 0.05$ at $\theta = 0$.　v_θ is the solution to (3·3·8) with $c_1 = 1$.

iii)　Viscous fluid

The second solution describes a swirling inflow of a viscous and compressible fluid.　There, an additional singularity emerges at $r = 1$.　Those singularities were avoided by placing a rotating boundary with some radial distance.　The details on this solution will be reported elsewhere.

Example 4 :　Unsteady flow

In this example, a time-dependent solution corresponding to the superposition of the wave functions is considered.　The simplest one may be a superposition of free plane waves :

$$\Psi = \sum_{j=1}^{N} \alpha_j e^{-i\omega_j t + i\boldsymbol{k}_j\cdot\boldsymbol{r}}, \quad \omega_j = \frac{\boldsymbol{k}_j^2}{2m}. \tag{3·4·1}$$

Here we set $N = 2$ and assume $\boldsymbol{k}_1 \times \boldsymbol{k}_2 \neq 0$ and α_j's are real.　By this specification of the wave function, the density and the phase are determined as

$$\rho_c = \alpha_1^2 + \alpha_2^2 + 2\alpha_1\alpha_2\cos(\omega t - \boldsymbol{k}\cdot\boldsymbol{r}), \quad \omega = \omega_1 - \omega_2 \neq 0, \quad \boldsymbol{k} = \boldsymbol{k}_1 - \boldsymbol{k}_2 \tag{3·4·2a}$$

$$\tan\delta = \frac{\sum_j \alpha_j \sin(\omega_j t - \boldsymbol{k}_j\cdot\boldsymbol{r})}{\sum_j \alpha_j \cos(\omega_j t - \boldsymbol{k}_j\cdot\boldsymbol{r})}. \tag{3·4·2b}$$

ρ_c and δ both are time-dependent.　Since

$$\nabla\rho_c = 2\alpha_1\alpha_2\boldsymbol{k}\sin(\omega t - \boldsymbol{k}\cdot\boldsymbol{r}) \tag{3·4·3a}$$

$$\nabla\delta = -\frac{1}{\rho_c}(\alpha_1^2\boldsymbol{k}_1 + \alpha_2^2\boldsymbol{k}_2 + \alpha_1\alpha_2\boldsymbol{K}\cos(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})), \quad \boldsymbol{K} = \boldsymbol{k}_1 + \boldsymbol{k}_2, \tag{3·4·3b}$$

(2·4·6) is written as

$$\boldsymbol{k}\cdot\nabla\ln\zeta' = \frac{a\boldsymbol{K}\cdot\boldsymbol{k}}{\nu} - \boldsymbol{k}\cdot\nabla\ln(\nu\sin(\omega t-\boldsymbol{k}\cdot\boldsymbol{r})) + \frac{\nabla X_z\times\boldsymbol{k}\,|_z}{\nu\rho_c}. \tag{3·4·4}$$

We assumed that the r dependence of ζ emerges through ρ_c. In order for ∇X_z to meet this condition, X_z must have the form like

$$X_z = \nu\rho_c\beta\boldsymbol{q}\cdot\boldsymbol{r} \tag{3·4·5}$$

β is an arbitrary function of $\omega t-\boldsymbol{k}\cdot\boldsymbol{r}$. In particular, β is allowed to be complex. This is possible because (3·4·4) is linear in ζ. X_z term gives a contribution in (3·4·4) only when $\boldsymbol{q}$ is not parallel to $\boldsymbol{k}$. Noting that $\nabla(\rho_c\beta)\propto\boldsymbol{k}$, we rewrite (3·4·4) as

$$\ln\left(\frac{\nu}{2\alpha_1\alpha_2\boldsymbol{k}^2}\boldsymbol{k}\cdot\nabla\zeta\right) = -\frac{a\boldsymbol{K}\cdot\boldsymbol{k}}{\nu\boldsymbol{k}^2}(\omega t-\boldsymbol{k}\cdot\boldsymbol{r}) - \frac{\boldsymbol{q}\times\boldsymbol{k}\,|_z}{\boldsymbol{k}^2}\int^{\omega t-\boldsymbol{k}\cdot\boldsymbol{r}} d\xi\beta(\xi). \tag{3·4·6}$$

Namely, ζ is an arbitrary function of $\omega t-\boldsymbol{k}\cdot\boldsymbol{r}$. If $\boldsymbol{q}=0$, the last term on the r.h.s. is absent and we would have a simple time-dependent extension of Example 2.

By way of example, we here consider a case

$$\zeta = c_1 e^{\boldsymbol{k}\cdot\boldsymbol{r}-\omega t},\quad A_z = \frac{c_1}{\boldsymbol{k}^2}e^{\boldsymbol{k}\cdot\boldsymbol{r}-\omega t} + \boldsymbol{V}\cdot\boldsymbol{r}, \tag{3·4·7a}$$

$$v_x = \frac{c_1 k_y}{\boldsymbol{k}^2}e^{\boldsymbol{k}\cdot\boldsymbol{r}-\omega t} + V_y,\quad v_y = -\frac{c_1 k_x}{\boldsymbol{k}^2}e^{\boldsymbol{k}\cdot\boldsymbol{r}-\omega t} - V_x, \tag{3·4·7b}$$

where $\boldsymbol{k}$, ω, $\boldsymbol{V}$ and c_1 are constant. Considering the arbitrariness of β, we allow these constants to be complex number. By choosing $\boldsymbol{q}=(-k_y, k_x, 0)$, the Navier-Stokes equation for the above $\boldsymbol{v}$ becomes

$$\frac{c_1\boldsymbol{q}}{\boldsymbol{k}^2}(\omega-k_xV_y+k_yV_x)e^{\boldsymbol{k}\cdot\boldsymbol{r}-\omega t} = -\nu c_1\boldsymbol{q}e^{\boldsymbol{k}\cdot\boldsymbol{r}-\omega t} - \frac{\nabla P}{\rho} + \boldsymbol{f}. \tag{3·4·8}$$

One of the reasonable assumptions for the density is that ρ varies as a function of $\omega t-\boldsymbol{k}\cdot\boldsymbol{r}$. In this case, the continuity equation (2·1·3) leads to the dispersion relation

$$\omega-k_xV_y+k_yV_x=0. \tag{3·4·9}$$

This means that the acceleration term of (2·1·1) identically vanishes and the viscous force, pressure gradient and the external force must be balanced by themselves. This condition is realized by

$$\rho^{-1}=\rho_0^{-1}+(c_1/c_2)\nu e^{\boldsymbol{k}\cdot\boldsymbol{r}-\omega t},\quad P=P_0-c_2\boldsymbol{q}\cdot\boldsymbol{r},\quad \boldsymbol{f}=-c_2\rho_0^{-1}\boldsymbol{q}, \tag{3·4·10}$$

where, ρ_0, P_0 and c_2 are constants. $\boldsymbol{f}$ given in (3·4·10) is also constant. As noted above, $\boldsymbol{k}$ and ω can be complex and physical quantities are obtained by taking the real parts in (3·4·7) and (3·4·10). Specifically, when $\boldsymbol{k}$ and ω are pure imaginary, the solution is a uniform propagating sound wave in a compressive fluid. The ‘sound’ velocity is $\boldsymbol{c} = \nabla_{\boldsymbol{k}} \cdot \omega = (V_y, -V_x, 0)$, where $\boldsymbol{V}$ has been assumed real. This is nothing but the average flow velocity. Namely, there is no propagation in the rest frame of the fluid.

If ρ is constant, it is possible to balance the pressure gradient term with the external force. In this case, it is easy to show, by assuming a general form $\zeta(\omega t - \boldsymbol{k} \cdot \boldsymbol{r})$ for ζ to determine A_z, that the velocity field is given by

$$v_x = \frac{c_1 k_y}{\boldsymbol{k}^2} e^{(-\omega + \boldsymbol{k} \times \boldsymbol{V}|_z)(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)/\nu \boldsymbol{k}^2} + V_y, \quad v_y = \frac{c_1 k_x}{\boldsymbol{k}^2} e^{(-\omega + \boldsymbol{k} \times \boldsymbol{V}|_z)(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)/\nu \boldsymbol{k}^2} + V_x. \qquad (3\cdot4\cdot11)$$

The continuity equation is satisfied without the constraint (3·4·9). In particular, nontrivial solutions are obtained even when $\boldsymbol{V} = 0$. The flow becomes the generalized Beltrami flow when $\boldsymbol{k} \times \boldsymbol{V}|_z = 0$. The case of $\boldsymbol{V} = 0$ has been studied by Taylor (1923), Kampe and Feriet (1930, 1932) and Wang (1966).

4. Two degrees of freedom

In the previous section, we assumed that ζ is a function of a single ρ_c and treated the differential equation in substantially one spatial dimension. In other words, we considered the flows of essentially one degree of freedom.

Owing to the linearity of (2·4·6) in ζ, the extension to the two degrees of freedom is, at least formally, straightforward. We adopt two sets of density and current which have independent coordinate dependences. Let the densities and phases be ρ_{cj} and the phases δ_j, $j = 1, 2$. Although some complexity emerges in (2·3·9) due to the coupling among ρ_{cj}’s, the equation will be simplified if one can choose two ρ_{cj} such that $\nabla\rho_{cj} \cdot \nabla\rho_{ck} = 0$ for $j \neq k$. In this case, (2·3·9) becomes a summation of the contribution from each ρ_{cj} and ζ will be expressed as $\zeta = \zeta_1(\rho_{c1}) + \zeta_2(\rho_{c2})$.

We apply the above idea to the second example in the previous section. Corresponding to two orthogonal vectors, we may have two independent ζ’s, which obey the equations

$$\ln \zeta_1' = -\frac{2ak_1}{\nu} x - \ln \partial_x \rho_{c1}, \quad \ln \zeta_2' = -\frac{2ak_2}{\nu} y - \ln \partial_y \rho_{c2} \qquad (4\cdot1)$$

The total ζ will be given by their sum as

$$\zeta = c_1 e^{c_2 x} + c_3 e^{c_4 y}. \qquad (4\cdot2)$$

The simplest stream function with no boundary may be

$$A_z = -\frac{c_1}{c_2^2} e^{c_2 x} - \frac{c_3}{c_4^2} e^{c_4 y} + c_5 x + c_6 y. \tag{4·3}$$

The velocity field is given by

$$v_x = -\frac{c_3}{c_4} e^{c_4 y} + c_6, \quad v_y = \frac{c_1}{c_2} e^{c_2 x} - c_5. \tag{4·4}$$

The continuity is satisfied for incompressible fluid. The Navier-Stoke equation is

$$\begin{aligned} -v_y c_3 e^{c_4 y} &= -\nu c_3 c_4 e^{c_4 y} - \frac{\partial_x P}{\rho} + f_x, \\ v_x c_1 e^{c_2 x} &= \nu c_1 c_2 e^{c_2 x} - \frac{\partial_y P}{\rho} + f_y. \end{aligned} \tag{4·5}$$

Comparing both sides, the consistent solution for the incompressible fluid is given by

$$c_2 = c_4 = -U/\nu, \quad c_5 = -c_6 = U. \tag{4·6}$$

This describes a flow 'into a corner' between semi-infinite planes having suction (Berker 1963).

5. Summary

Focusing on the two-dimensional solenoidal flows, we derived a liner differential equation – the transmutation equation – that relates the vorticity ζ to arbitrary conserved currents. The stream function A_z is obtained by solving Poisson's equation with ζ as source function. Any conserved currents will be used as inputs to the transmutation equation to obtain numerical solutions.

Some exact solutions are also obtained through the method, which shows the consistency of the transmutation equation with the Navier-Stokes equation. The integration constants that are introduced in this procedure are, together with the density, the pressure and the external force, determined from the requirement that the velocity field obeys the original Navier-Stokes equation and the continuity equation. In this paper, this matching process was shown to be performed easily and consistently.

The transmutation equation involves the X_z term that emerges when the combined equation of the vorticity and continuity equations is integrated. We saw that the familiar solutions were obtained in case the X_z term was neglected. When X_z term was pertinently taken into account, interesting new solutions were found. In particular, the solution for the inviscid concentric flow reproduces the profile of the horizontal air flow of tropical cyclone or typhoon extremely well (Emanuel 2004, Holland et al. 2010, Takahashi 2012).

Our method facilitates solving the Navier–Stoke equation for one degree of freedom, and will be exploitable in two degrees of freedom, too.　Whether an extension to the three–dimension is possible is an open question.

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Appendix

Here we show that, when the right hand sides of (3·3·9) are neglected, δv_r is small as compared to δv_θ for high modes.　Take the time derivative of (3·3·9b) and substitute (3·3·9a) to eliminate $\delta \dot{v}_r$:

$$\delta \ddot{v}_\theta + \frac{v_\theta}{r}\partial_\theta \delta \dot{v}_\theta + \frac{2v_\theta}{r}\left(\partial_r v_\theta + \frac{v_\theta}{r}\right)\delta v_\theta = 0, \tag{A1}$$

where v_θ is a function of r only.　Substituting for δv_θ an ansatz

$$\delta v_\theta = \Psi\left(g(r)(t-t_0)-\theta\right) \tag{A2}$$

where t_0 is a constant, we have

$$\left(g(r)^2 - \frac{v_\theta}{r}g(r)\right)\Psi''(\eta) + \frac{2v_\theta}{r}\left(\partial_r v_\theta + \frac{v_\theta}{r}\right)\Psi(\eta) = 0. \tag{A3}$$

Here, primes on ψ stand for derivatives with respect to $\eta \equiv g(r)(t-t_0)-\theta$.　In order for (A3) to have non–trivial solutions, the ratio of the coefficients of ψ and Ψ'' must be a constant, n^2.　Then, (A3) decomposes to two equations :

$$g(r)^2 - \frac{v_\theta}{r}g(r) + \frac{2v_\theta}{n^2 r}\left(\partial_r v_\theta + \frac{v_\theta}{r}\right) = 0, \tag{A4}$$

$$\Psi''(\eta) + n^2\,\Psi(\eta) = 0. \tag{A5}$$

(A5) implies that $\Psi(\eta)$ is a sinusoidal function of η,

$$\Psi = \sin n\eta \ \text{ or } \ \cos n\eta. \tag{A6}$$

Since δv_θ is single–valued in θ, n must be an integer.

(A4) yields

$$g(r)=\frac{1}{2}\left[\frac{v_\theta}{r}\pm\sqrt{\left(1+\frac{8}{n^2}\right)\left(\frac{v_\theta}{r}\right)^2+\frac{8v_\theta v'_\theta}{n^2 r}}\right]. \tag{A7}$$

For time-dependent solutions with high mode, $g(r)\sim v_\theta/r$. In this case, (3·3·9a) together with (A2) and (3·3·9b) results in

$$\delta v_r \sim -\frac{2}{n}\cos n\eta. \tag{A8}$$

This proves the smallness of v_r relative to v_θ for large n.

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